Function Theory

$\langle v: D \rightarrow b \rangle$ "map v in D to b"

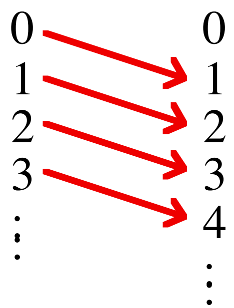
body (may use v)

domain, type (a bunch)

variable, parameter (a fresh name)

$v: D$ is a local axiom within b

$\langle n: nat \rightarrow n{+}1 \rangle$



Renaming

$\langle n: nat \rightarrow n+1 \rangle \quad = \quad \langle m: nat \rightarrow m+1 \rangle$

Domain

Δf "domain of f"

$\Delta \langle n: nat \rightarrow n+1 \rangle \quad = \quad nat$

Application

$f\, x$ " f applied to x " or " f of x "

$f\,(x) \qquad (f)x \qquad f\,(x+1) \qquad -x \qquad \neg x$

$\langle n: nat \rightarrow n+1 \rangle\, 3 \;=\; 3+1 \;=\; 4$

two variables

$max\ =\ \langle x{:}\ xrat \rightarrow \langle y{:}\ xrat \rightarrow \mathbf{if}\ x{\ge}y\ \mathbf{then}\ x\ \mathbf{else}\ y\rangle\rangle$

$max\ 3\ =\ \langle y{:}\ xrat \rightarrow \mathbf{if}\ 3{\ge}y\ \mathbf{then}\ 3\ \mathbf{else}\ y\rangle$

$max\ 3\ 5\ =\ (\mathbf{if}\ 3{\ge}5\ \mathbf{then}\ 3\ \mathbf{else}\ 5)\ =\ 5$

$max(3, 5)\ =\ max\ 3, max\ 5$

predicate

function with boolean result

$even\ =\ \langle i{:}\ int \rightarrow i/2{:}\ int\rangle$

relation

function with predicate result

$divides\ =\ \langle n{:}\ nat{+}1 \rightarrow \langle i{:}\ int \rightarrow i/n{:}\ int\rangle\rangle$

$even\ =\ divides\ 2$

selective union

$f \mid g$ "f otherwise g"

$\Delta(f \mid g) \;=\; \Delta f, \Delta g$

$(f \mid g)\, x \;=\; \textbf{if } x\text{: } \Delta f \textbf{ then } f\, x \textbf{ else } g\, x$

abbreviated function notations

$\langle x\text{: } xrat \rightarrow \langle y\text{: } xrat \rightarrow \textbf{if } x \ge y \textbf{ then } x \textbf{ else } y\rangle\rangle \;=\; \langle x, y\text{: } xrat \rightarrow \textbf{if } x \ge y \textbf{ then } x \textbf{ else } y\rangle$

$\langle n\text{: } nat \rightarrow n+1\rangle \;=\; \langle n \rightarrow n+1\rangle$

$\langle n\text{: } 2 \rightarrow 3\rangle \;=\; 2 \rightarrow 3$ scope brackets go with variable

$\langle x\text{: } int \rightarrow \langle y\text{: } int \rightarrow x+3\rangle\rangle \;=\; x+3$? but we can't apply it

Scope and Substitution

local bound, hidden, private

introduction is inside the expression (formal)

nonlocal global, free, visible, public

introduction is outside the expression (formal or informal)

$$\langle x \rightarrow \quad x \quad y \quad \rangle (\quad x \quad y \quad)$$

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$$\langle x \rightarrow \quad x \quad \langle x \rightarrow \quad x \quad \rangle \quad x \quad \rangle\, 3$$

$$= \quad (\quad 3 \quad \langle x \rightarrow \quad x \quad \rangle \quad 3 \quad)$$

Scope and Substitution

local bound, hidden, private

introduction is inside the expression (formal)

nonlocal global, free, visible, public

introduction is outside the expression (formal or informal)

$$\langle y \rightarrow \quad x \quad y \quad \langle x \rightarrow \quad x \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$$ rename inner x to z

$$= \quad \langle y \rightarrow \quad x \quad y \quad \langle z \rightarrow \quad z \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$$ now apply

$$= \quad (\qquad x \quad x \quad \langle z \rightarrow \quad z \quad x \quad \rangle \quad x \quad x \quad)$$

Quantifiers

A quantifier is an operator that applies to a function.

It is defined from a two-operand symmetric associative operator.

$\forall p$ is defined from $\wedge$	$\forall\langle r\colon rat \rightarrow r<0 \vee r=0 \vee r>0\rangle$
$\exists p$ is defined from $\vee$	$\exists\langle n\colon nat \rightarrow n=0\rangle$
Σf is defined from $+$	$\Sigma\langle n\colon nat+1 \rightarrow 1/2^n\rangle$
Πf is defined from $\times$	$\Pi\langle n\colon nat+1 \rightarrow (4\times n^2)/(4\times n^2-1)\rangle$

abbreviations

$\forall r\colon rat\cdot\ r<0 \vee r=0 \vee r>0$

$\Sigma n\colon nat+1\cdot\ 1/2^n$

$\forall x, y\colon rat\cdot\ x = y+1 \ \Rightarrow\ x > y \quad = \quad \forall x\colon rat\cdot\ \forall y\colon rat\cdot\ x = y+1 \ \Rightarrow\ x > y$

$\Sigma n, m\colon 0,..10\cdot\ n\times m \quad = \quad \Sigma n\colon 0,..10\cdot\ \Sigma m\colon 0,..10\cdot\ n\times m$

$\forall v\colon null\cdot\ b \ = \ \top$

$\forall v\colon x\cdot\ b \ = \ \langle v\colon x \to b\rangle\ x$

$\forall v\colon A,B\cdot\ b \ = \ (\forall v\colon A\cdot\ b) \land (\forall v\colon B\cdot\ b)$

$\exists v\colon null\cdot\ b \ = \ \bot$

$\exists v\colon x\cdot\ b \ = \ \langle v\colon x \to b\rangle\ x$

$\exists v\colon A,B\cdot\ b \ = \ (\exists v\colon A\cdot\ b) \lor (\exists v\colon B\cdot\ b)$

$\Sigma v\colon null\cdot\ n \ = \ 0$

$\Sigma v\colon x\cdot\ n \ = \ \langle v\colon x \to n\rangle\ x$

$(\Sigma v\colon A,B\cdot\ n) + (\Sigma v\colon A\text{‘}B\cdot\ n) \ = \ (\Sigma v\colon A\cdot\ n) + (\Sigma v\colon B\cdot\ n)$

$\Pi v\colon null\cdot\ n \ = \ 1$

$\Pi v\colon x\cdot\ n \ = \ \langle v\colon x \to n\rangle\ x$

$(\Pi v\colon A,B\cdot\ n) \times (\Pi v\colon A\text{‘}B\cdot\ n) \ = \ (\Pi v\colon A\cdot\ n) \times (\Pi v\colon B\cdot\ n)$

build your own

$$MAX\ \ x\colon rat\cdot\ 4{\times}x - x^2\ \ =\ \ 4$$

$$MAX\ v\colon null\cdot\ n\ \ =\ \ -\infty$$

$$MAX\ v\colon x\cdot\ n\ \ =\ \ \langle v\colon x \rightarrow n\rangle\ x$$

$$MAX\ v\colon A{,}B\cdot\ n\ \ =\ \ max\ (MAX\ v\colon A\cdot\ n)\ (MAX\ v\colon B\cdot\ n)$$

$$x \wedge \top\ \ =\ \ x$$

$$x \vee \bot\ \ =\ \ x$$

$$x + 0\ \ =\ \ x$$

$$x \times 1\ \ =\ \ x$$

$$max\ x\ (-\infty)\ \ =\ \ x$$

Solution Quantifier

$\S p$ is the (bunch of) solutions of predicate p

$\S v\text{: } null\cdot\ b \;=\; null$

$\S v\text{: } x\cdot\ b \;=\; \textbf{if}\ \langle v\text{: } x \rightarrow b\rangle\ x\ \textbf{then}\ x\ \textbf{else}\ null$

$\S v\text{: } A{,}B\cdot\ b \;=\; (\S v\text{: } A\cdot\ b), (\S v\text{: } B\cdot\ b)$

$\{\ \S i\text{: } int\cdot\ i^2{=}4\ \} \;=\; \{-2, 2\ \}$

$\{\ \S n\text{: } nat\cdot\ n{<}3\ \} \;=\; \{0,..3\ \}$

An expression talks about its nonlocal variables.

$$\exists n: nat \cdot x = 2 \times n$$

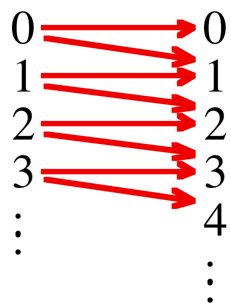
says

"x is an even natural"

Function Fine Points

partial:	sometimes no result
total:	always at least one result
deterministic:	always at most one result
nondeterministic:	sometimes more than one result

$\langle n\text{: } nat\cdot\ n, n{+}1\rangle$ total and nondeterministic



$\langle n\text{: } nat\cdot\ n, n{+}1\rangle\ 3 \ = \ 3, 4$

distribution

$$(f, g)\, x \;=\; fx, gx$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\colon nat \rightarrow n{+}n \rangle$$

$$double\ 2 \;=\; 4$$

$$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$$

$$double\ (2, 3) \;\neq\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

$$tiny \;=\; \langle S\colon \natural nat \rightarrow \$S < 3 \rangle$$

$$tiny\ \{null\} \;=\; \top$$

$$tiny\ \{0, 1, 2, 3\} \;=\; \bot$$

$$tiny\ null \;=\; null$$

function inclusion

$$f: g \;=\; \Delta g: \Delta f \;\land\; \forall x: \Delta g\cdot\ fx: gx$$

$$A\to B \;=\; \langle a: A \to B\rangle$$

$A\to B$ is a function whose domain is A and whose result is B .

$$f: A\to B \;=\; A: \Delta f \;\land\; \forall a: A\cdot\ fa: B$$

$A\to B$ is all those functions whose domain is at least A and whose result is at most B .

$A\to B$ is antimonotonic in A and monotonic in B .

	$suc: nat\to nat$	function inclusion
$=$	$nat: \Delta suc \;\land\; \forall n: nat\cdot\ suc\ n: nat$	definition of suc
$=$	$nat: nat \;\land\; \forall n: nat\cdot\ n{+}1: nat$	reflexivity and definition of nat
$=$	$\top$	

function inclusion

$f: g \quad = \quad \Delta g: \Delta f \;\land\; \forall x: \Delta g \cdot fx: gx$

$suc: nat \to nat$

$even: int \to bool$

$max: xrat \to xrat \to xrat$

$A: B \quad \land \quad f: B \to C \quad \land \quad C: D \quad \Rightarrow \quad f: A \to D$

$(0,..10): nat \quad \land \quad suc: nat \to nat \quad \land \quad nat: int \quad \Rightarrow \quad suc: (0,..10) \to int$

$\langle f: (0,..10) \to int \cdot \forall n: 0,..10 \cdot even\ (f\ n) \rangle\ suc$

$= \quad \forall n: 0,..10 \cdot even\ (suc\ n)$

$= \quad \bot$

function equality

$$f = g \;=\; \Delta f = \Delta g \;\wedge\; \forall x\colon \Delta f \cdot fx = gx$$

function composition

If $\neg f: \Delta g$ then

$$\Delta(g\ f) = \S x: \Delta f \cdot f x: \Delta g$$

$$(g\ f)\ x = g\ (f\ x)$$

$$\Delta(even\ suc)$$

$$= \S x: \Delta suc \cdot suc\ x: \Delta even$$

$$= \S x: nat \cdot x+1: int$$

$$= nat$$

$$(even\ suc)\ 3 = even\ (suc\ 3) = even\ 4 = \top$$

$$(-suc)\ 3 = -(suc\ 3) = -4$$

$$(\neg even)\ 3 = \neg(even\ 3) = \neg\bot = \top$$

function composition

Suppose x, y: *int*

f, g: *int*→*int*

h: *int*→*int*→*int*

Then

$$h\ f\ x\ g\ y$$

$$= \quad (((h\ f)\ x)\ g)\ y$$

$$= \quad ((h\ (f\ x))\ g)\ y$$

$$= \quad (h\ (f\ x))\ (g\ y)$$

$$= \quad h\ (f\ x)\ (g\ y)$$

list as function

If m: 0,..#L then

$\langle n: 0,..\#L \rightarrow Ln\rangle\ m \quad = \quad L\ m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

$-[3; 5; 2] \quad = \quad [-3; -5; -2]$

$suc\ [3; 5; 2] \quad = \quad [4; 6; 3]$

$1 \rightarrow 21 \mid [10; 11; 12] \quad = \quad [10; 21; 12]$

$\Sigma L \quad = \quad \Sigma n: 0,..\#L \cdot Ln$

limit

f: $nat \rightarrow rat$

$f0; f1; f2; ...$ is a sequence of rationals

$(MAX\ m\cdot\ MIN\ n\cdot\ f(m+n)) \leq (LIM\ f) \leq (MIN\ m\cdot\ MAX\ n\cdot\ f(m+n))$

$LIM\ n\cdot\ 1/(n+1) \quad = \quad 0$

$-1 \leq (LIM\ n\cdot\ (-1)^n) \leq 1$

$(MIN\ f) \leq (LIM\ f) \leq (MAX\ f)$

$xreal \quad = \quad LIM\ (nat \rightarrow rat)$

limit

p: $nat \rightarrow bool$

$p0; p1; p2; \ldots$ is a sequence of booleans

$$\exists m \cdot \forall n \cdot p(m+n) \quad \Longrightarrow \quad LIM\ p \quad \Longrightarrow \quad \forall m \cdot \exists n \cdot p(m+n)$$

$$\exists m \cdot \forall i \cdot i \ge m \Rightarrow pi \quad \Longrightarrow \quad LIM\ p$$

$$\exists m \cdot \forall i \cdot i \ge m \Rightarrow \neg pi \quad \Longrightarrow \quad \neg\ LIM\ p$$

$$LIM\ n \cdot 1/(n+1) = 0 \quad = \quad \bot$$