Recursive Data Definition

example: *nat*

can be constructed by starting with 0 and repeatedly adding 1

construction axiom $\quad$ 0: *nat*

construction axiom $\quad$ *nat*+1: *nat*

	⊤	by the axiom, 0: *nat*
⇒	0: *nat*	add 1 to each side
⇒	0+1: *nat*+1	by arithmetic, 0+1 = 1 ; by the axiom, *nat*+1: *nat*
⇒	1: *nat*	add 1 to each side
⇒	1+1: *nat*+1	by arithmetic, 1+1 = 2 ; by the axiom, *nat*+1: *nat*
⇒	2: *nat*	and so on

Recursive Data Definition

example: *nat*

can be constructed by starting with 0 and repeatedly adding 1

construction axiom   0: *nat*

construction axiom   *nat*+1: *nat*

nat = 0, 1, 2, 3, 4, 5, ... ?

nat = ..., -3, -2, -1, 0, 1, 2, 3, ... ?

nat = the rationals ?

nat = the reals ?

nat = 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, ... ?

Recursive Data Definition

example: *nat*

can be constructed by starting with 0 and repeatedly adding 1

construction axiom	$0: nat$
construction axiom	$nat+1: nat$
induction axiom	$0: B \land B+1: B \Rightarrow nat: B$

construction axiom	$0, nat+1: nat$
induction axiom	$0, B+1: B \Rightarrow nat: B$

construction axiom	$P0 \land \forall n: nat \cdot Pn \Rightarrow P(n+1) \Leftarrow \forall n: nat \cdot Pn$
induction axiom	$P0 \land \forall n: nat \cdot Pn \Rightarrow P(n+1) \Rightarrow \forall n: nat \cdot Pn$

Recursive Data Definition

nat induction

$P0 \wedge \forall n: nat\cdot\ Pn \Rightarrow P(n+1) \quad \Longrightarrow \quad \forall n: nat\cdot\ Pn$

$P0 \vee \exists n: nat\cdot\ \neg Pn \wedge P(n+1) \quad \Longleftarrow \quad \exists n: nat\cdot\ Pn$

$\forall n: nat\cdot\ Pn \Rightarrow P(n+1) \quad \Longrightarrow \quad \forall n: nat\cdot\ (P0 \Rightarrow Pn)$

$\exists n: nat\cdot\ \neg Pn \wedge P(n+1) \quad \Longleftarrow \quad \exists n: nat\cdot\ (\neg P0 \wedge Pn)$

$\forall n: nat\cdot\ (\forall m: nat\cdot\ m<n \Rightarrow Pm) \Rightarrow Pn \quad \Longrightarrow \quad \forall n: nat\cdot\ Pn$

$\exists n: nat\cdot\ (\forall m: nat\cdot\ m<n \Rightarrow \neg Pm) \wedge Pn \quad \Longleftarrow \quad \exists n: nat\cdot\ Pn$

philosophical induction: guessing the general case from special cases

(an important skill in mathematics)

philosophical deduction: proving, using the rules of logic

mathematical induction: an axiom (sometimes presented as a proof rule)

(mathematical induction is part of philosophical deduction)

Recursive Data Definition

example: *int*

Define $\quad int \;=\; nat, -nat$

or $\quad 0, int+1, int-1: int$

$0, B+1, B-1: B \;\Rightarrow\; int: B$

or $\quad P0 \wedge (\forall i: int\cdot Pi \Rightarrow P(i+1)) \wedge (\forall i: int\cdot Pi \Rightarrow P(i-1)) \;=\; \forall i: int\cdot Pi$

Recursive Data Definition

example: *pow*

Define $pow \;=\; 2^{nat}$

or $pow \;=\; \S p: nat\cdot\ \exists m: nat\cdot\ p = 2^m$

or $1, 2\times pow: pow$

$1, 2\times B: B \;\Rightarrow\; pow: B$

or $P1 \;\wedge\; \forall p: pow\cdot\ Pp \Rightarrow P(2\times p) \;=\; \forall p: pow\cdot\ Pp$

Least Fixed-Points

nat construction:	$0, nat+1: nat$
nat induction:	$0, B+1: B \;\Rightarrow\; nat: B$
nat fixed-point construction:	$nat \;=\; 0, nat+1$
nat fixed-point induction:	$B = 0, B+1 \;\Rightarrow\; nat: B$
x is a fixed-point of f	$x \;=\; f\, x$
grammar:	$exp \;=\;$ "x", exp; "+"; exp
	$B \;=\;$ "x", B; "+"; $B \;\Rightarrow\; exp: B$

Recursive Data Construction

$name \; = \;$ (expression involving $\; name$)

0. Construct — $name_0 \; = \; null$

   $name_{n+1} \; = \;$ (expression involving $\; name_n$)

1. Guess — $name_n \; = \;$ (expression involving $\; n \;$ but not $\; name$)

2. Substitute $\; \infty \;$ for $\; n$ — $name_\infty \; = \;$ (expression involving neither $\; n \;$ nor $\; name$)

3. Test fixed-point — $name_\infty \; = \;$ (expression involving $\; name_\infty$)

4. Test least fixed-point — $B =$ (expression involving $\; B$) $\; \Rightarrow \; name_\infty\!: B$

Recursive Data Construction

example: pow

$$pow \quad = \quad 1, 2{\times}pow$$

0. Construct

$$pow_0 \quad = \quad null$$

$$pow_1 \quad = \quad 1, 2{\times}pow_0 \quad = \quad 1, 2{\times}null \quad = \quad 1, null \quad = \quad 1$$

$$pow_2 \quad = \quad 1, 2{\times}pow_1 \quad = \quad 1, 2{\times}1 \quad = \quad 1, 2$$

$$pow_3 \quad = \quad 1, 2{\times}pow_2 \quad = \quad 1, 2{\times}(1, 2) \quad = \quad 1, 2, 4$$

1. Guess $\qquad pow_n \; = \; 2^{0,..n}$

2. Substitute ∞ for n $\qquad pow_\infty \; = \; 2^{0,..\infty} \; = \; 2^{nat}$

Recursive Data Construction

example: *pow*

$$pow \;=\; 1, 2{\times}pow$$

3. Test fixed-point.

$$2^{nat} = 1, 2{\times}2^{nat}$$

$$= \quad 2^{nat} = 2^0, 2^1{\times}2^{nat}$$

$$= \quad 2^{nat} = 2^0, 2^{1+nat}$$

$$= \quad 2^{nat} = 2^{0,\ 1+nat}$$

$$\Leftarrow \quad nat = 0, nat+1$$

$$= \quad \top$$

Recursive Data Construction

example: *pow*

$pow \ = \ 1, 2 \times pow$

4. Test least fixed-point

	$2^{nat}: B$	
$=$	$\forall n: nat \cdot 2^n: B$	use *nat* induction with $Pn = 2^n: B$
$\Leftarrow$	$2^0: B \ \land \ \forall n: nat \cdot \ 2^n: B \Rightarrow 2^{n+1}: B$	change variable
$=$	$1: B \ \land \ \forall m: 2^{nat} \cdot \ m: B \Rightarrow 2 \times m: B$	increase domain
$\Leftarrow$	$1: B \ \land \ \forall m: nat \cdot \ m: B \Rightarrow 2 \times m: B$	domain change law
$=$	$1: B \ \land \ \forall m: nat`B \cdot 2 \times m: B$	increase domain
$\Leftarrow$	$1: B \ \land \ \forall m: B \cdot 2 \times m: B$	
$\Leftarrow$	$B = 1, 2 \times B$	

Recursive Data Construction

Alternative step 0: instead of *null* use

$name_0 = whatever$

Alternative step 2: instead of $name_\infty$ use

$\S x \cdot LIM\ n \cdot x : name_n$

Recursive Specification Definition

$zap \;=\; \textbf{if } x{=}0 \textbf{ then } y{:=}\,0 \textbf{ else } (x{:=}\,x{-}1.\; t{:=}\,t{+}1.\; zap)$

solutions

(a) $x{\ge}0 \;\Rightarrow\; x'{=}y'{=}0 \land t' = t{+}x$

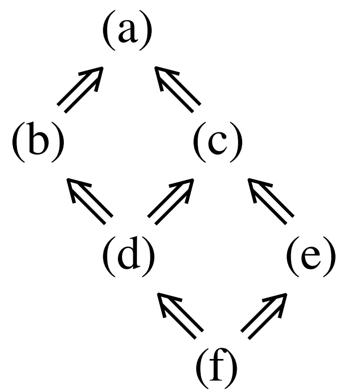
(b) $\textbf{if } x{\ge}0 \textbf{ then } x'{=}y'{=}0 \land t' = t{+}x \textbf{ else } t'{=}\infty$

(c) $x'{=}y'{=}0 \land (x{\ge}0 \Rightarrow t' = t{+}x)$

(d) $x'{=}y'{=}0 \land \textbf{if } x{\ge}0 \textbf{ then } t' = t{+}x \textbf{ else } t'{=}\infty$

(e) $x'{=}y'{=}0 \land t' = t{+}x$

(f) $x{\ge}0 \land x'{=}y'{=}0 \land t' = t{+}x$



$x{\ge}0 \;\Rightarrow\; x'{=}y'{=}0 \land t' = t{+}x \;\Leftarrow\; zap$

$zap \;\Leftarrow\; \textbf{if } x{=}0 \textbf{ then } y{:=}\,0 \textbf{ else } (x{:=}\,x{-}1.\; t{:=}\,t{+}1.\; zap)$

Recursive Specification Definition

zap construction

$$t' \geq t \;\Leftarrow\; zap$$

$$\textbf{if } x=0 \textbf{ then } y:= 0 \textbf{ else } (x:= x-1.\; t:= t+1.\; zap) \;\Leftarrow\; zap$$

nat construction

$$0: nat$$

$$nat+1: nat$$

Recursive Specification Definition

zap construction

$$t' \geq t \land \textbf{if } x{=}0 \textbf{ then } y{:=}\,0 \textbf{ else } (x{:=}\,x{-}1.\;\; t{:=}\,t{+}1.\;\; zap) \;\Leftarrow\; zap$$

zap induction

$$\forall\sigma, \sigma'\cdot\; t' \geq t \land (\textbf{if } x{=}0 \textbf{ then } y{:=}\,0 \textbf{ else } (x{:=}\,x{-}1.\;\; t{:=}\,t{+}1.\;\; P)) \;\Leftarrow\; P$$

$$\Longrightarrow \quad \forall\sigma, \sigma'\cdot\; zap \Leftarrow P$$

nat construction

$$0,\, nat{+}1\colon nat$$

nat induction

$$0,\, B{+}1\colon B \;\;\Longrightarrow\;\; nat\colon B$$

Recursive Specification Definition

zap construction

$$t' \geq t \wedge \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap) \ \Leftarrow\ zap$$

zap induction

$$\forall \sigma, \sigma' \cdot\ t' \geq t \wedge (\textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ P)) \ \Leftarrow\ P$$

$$\Rightarrow \quad \forall \sigma, \sigma' \cdot\ zap \Leftarrow P$$

zap fixed-point construction

$$zap \ = \ t' \geq t \wedge \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap)$$

zap fixed-point induction

$$\forall \sigma, \sigma' \cdot\ (P \ = \ t' \geq t \wedge \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ P))$$

$$\Rightarrow \quad \forall \sigma, \sigma' \cdot\ zap \Leftarrow P$$

Recursive Specification Construction

$zap \quad = \quad$ **if** $x{=}0$ **then** $y{:=}\ 0$ **else** $(x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap)$

$zap_0 \quad = \quad \top$

$zap_1 \quad = \quad$ **if** $x{=}0$ **then** $y{:=}\ 0$ **else** $(x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap_0)$

$\quad = \quad x{=}0 \ \Rightarrow\ x'{=}y'{=}0 \wedge t'{=}t$

$zap_2 \quad = \quad$ **if** $x{=}0$ **then** $y{:=}\ 0$ **else** $(x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap_1)$

$\quad = \quad 0{\le}x{<}2 \ \Rightarrow\ x'{=}y'{=}0 \wedge t' = t{+}x$

$zap_n \quad = \quad 0{\le}x{<}n \ \Rightarrow\ x'{=}y'{=}0 \wedge t' = t{+}x$

$zap_\infty \quad = \quad 0{\le}x{<}\infty \ \Rightarrow\ x'{=}y'{=}0 \wedge t' = t{+}x$

Recursive Specification Construction

Alternative step 0: instead of $\top$ use

$name_0 = whatever$

Alternative step 2: instead of $name_\infty$ use

$LIM\ n\cdot\ name_n$

Recursive Specification Construction

$$zap = \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap)$$

$$zap_0 = t' \ge t$$

$$zap_1 = \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap_0)$$

$$= \textbf{if } x{=}0 \textbf{ then } x'{=}y'{=}0 \land t'{=}t \textbf{ else } t' \ge t{+}1$$

$$zap_2 = \textbf{if } x{=}0 \textbf{ then } y{:=}\ 0 \textbf{ else } (x{:=}\ x{-}1.\ \ t{:=}\ t{+}1.\ \ zap_1)$$

$$= \textbf{if } 0{\le}x{<}2 \textbf{ then } x'{=}y'{=}0 \land t' = t{+}x \textbf{ else } t' \ge t{+}2$$

$$zap_n = \textbf{if } 0{\le}x{<}n \textbf{ then } x'{=}y'{=}0 \land t'{=}t{+}x \textbf{ else } t' \ge t{+}n$$

$$zap_\infty = \textbf{if } 0{\le}x \textbf{ then } x'{=}y'{=}0 \land t'{=}t{+}x \textbf{ else } t'{=}\infty$$

Loop Definition

while-loop construction

$t' \ge t \quad \Leftarrow \quad$ **while** b **do** P

if b **then** $(P.\ \ t{:=}\ t{+}1.$ **while** b **do** $P)$ **else** $ok \quad \Leftarrow \quad$ **while** b **do** P

Loop Definition

while-loop construction

$$t' \geq t \land \textbf{if } b \textbf{ then } (P.\ \ t{:=}\ t{+}1.\ \ \textbf{while } b \textbf{ do } P) \textbf{ else } ok \ \Leftarrow\ \textbf{while } b \textbf{ do } P$$

while-loop induction

$$\forall \sigma, \sigma' \cdot\ t' \geq t \land (\textbf{if } b \textbf{ then } (P.\ \ t{:=}\ t{+}1.\ \ W) \textbf{ else } ok) \Leftarrow W$$

$$\Rightarrow \quad \forall \sigma, \sigma' \cdot\ (\textbf{while } b \textbf{ do } P) \Leftarrow W$$

while-loop fixed-point construction

$$\textbf{while } b \textbf{ do } P \ = \ t' \geq t \land \textbf{if } b \textbf{ then } (P.\ \ t{:=}\ t{+}1.\ \ \textbf{while } b \textbf{ do } P) \textbf{ else } ok$$

while-loop fixed-point induction

$$\forall \sigma, \sigma' \cdot\ (P \ = \ t' \geq t \land \textbf{if } b \textbf{ then } (P.\ \ t{:=}\ t{+}1.\ \ W) \textbf{ else } ok)$$

$$\Rightarrow \quad \forall \sigma, \sigma' \cdot\ (\textbf{while } b \textbf{ do } P) \Leftarrow W$$