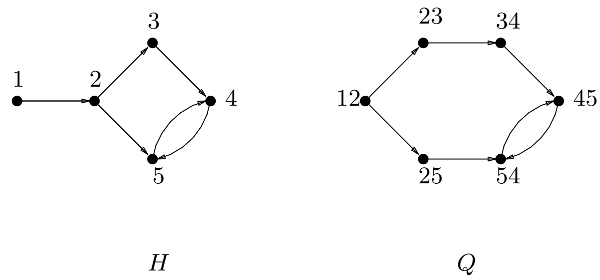

Figure 4.4 A digraph H and its line digraph $Q = L(H)$.

[419] by Beineke and Hemminger. The proof presented here is adapted from [419]. For an $n \times n$-matrix $M = [m_{ik}]$, a row i is **orthogonal** to a row j if $\sum_{k=1}^{n} m_{ik}m_{jk} = 0$. One can give a similar definition of orthogonal columns.

Theorem 4.5.1 *Let D be a directed pseudograph with vertex set $\{1, 2, \ldots, n\}$ and with no parallel arcs and let $M = [m_{ij}]$ be its adjacency matrix (i.e., the $n \times n$-matrix such that $m_{ij} = 1$, if $ij \in A(D)$, and $m_{ij} = 0$, otherwise). Then the following assertions are equivalent:*

(i) D is a line digraph;
(ii) there exist two partitions $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ of $V(D)$ such that $A(D) = \cup_{i \in I} A_i \times B_i$[4];
(iii) if vw, uw and ux are arcs of D, then so is vx;
(iv) any two rows of M are either identical or orthogonal;
(v) any two columns of M are either identical or orthogonal.

Proof: We show the following implications and equivalences: (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Leftrightarrow$ (v), (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii). Let $D = L(H)$. For each $v_i \in V(H)$, let A_i and B_i be the sets of in-coming and out-going arcs at v_i, respectively. Then the arc set of the subdigraph of D induced by $A_i \cup B_i$ equals $A_i \times B_i$. If $ab \in A(D)$, then there is an i such that $a = v_jv_i$ and $b = v_iv_k$. Hence, $ab \in A_i \times B_i$. The result follows.

(ii) $\Rightarrow$ (i). Let Q be the directed pseudograph with ordered pairs (A_i, B_i) as vertices, and with $|A_j \cap B_i|$ arcs from (A_i, B_i) to (A_j, B_j) for each i and j (including $i = j$). Let σ_{ij} be a bijection from $A_j \cap B_i$ to this set of arcs (from (A_i, B_i) to (A_j, B_j)) of Q. Then the function σ defined on $V(D)$ by taking σ to be σ_{ij} on $A_j \cap B_i$ is a well-defined function of $V(D)$ into $V(L(Q))$, since $\{A_j \cap B_i\}_{i,j \in I}$ is a partition of $V(D)$. Moreover, σ is a bijection since every σ_{ij} is a bijection. Furthermore, it is not difficult to see that σ is an isomorphism from D to $L(Q)$ (this is left as Exercise 4.4).

[4] Recall that $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

(ii) $\Rightarrow$ (iii). If vw, uw and ux are arcs of D, then there exist i, j such that $\{u, v\} \subseteq A_i$ and $\{w, x\} \subseteq B_j$. Hence, $(v, x) \in A_i \times B_j$ and $vx \in D$.

(iii) $\Rightarrow$ (iv). Assume that (iv) does no hold. This means that some rows, say i and j, are neither identical nor orthogonal. Then there exist k, h such that $m_{ik} = m_{jk} = 1$ and $m_{ih} = 1, m_{jh} = 0$ (or vice versa). Hence, ik, jk, ih are in $A(D)$ but jh is not. This contradicts (iii).

(iv) $\Leftrightarrow$ (v). Both (iv) and (v) are equivalent to the statement:

$$\text{for all } i, j, h, k, \text{ if } m_{ih} = m_{ik} = m_{jk} = 1, \text{ then } m_{jh} = 1.$$

(iv) $\Rightarrow$ (ii). For each i and j with $m_{ij} = 1$, let $A_{ij} = \{h : m_{hj} = 1\}$ and $B_{ij} = \{k : m_{ik} = 1\}$. Then, by (iv), A_{ij} is the set of vertices in D whose row vectors in M are identical to the ith row vector, whereas B_{ij} is the set of vertices in D whose column vectors in M are identical to the jth column vector (we use the previously proved fact that (iv) and (v) are equivalent). Thus, $A_{ij} \times B_{ij} \subseteq A(D)$, and moreover $A(D) = \cup\{A_{ij} \times B_{ij} : m_{ij} = 1\}$. By the orthogonality condition, A_{ij} and A_{hk} are either equal or disjoint, as are B_{ij} and B_{hk}. For zero row vector i in M, let A_{ij} be the set of vertices whose row vector in M is the zero vector, and let $B_{ij} = \emptyset$. Doing the same with the zero column vectors of M completes the partition as in (ii). $\square$

The characterizations (ii)-(v) all imply polynomial algorithms to verify whether a given directed pseudograph is a line digraph. This fact is obvious regarding (iii)-(v); it is slightly more difficult to see that (ii) can be used to construct a very effective polynomial algorithm. We actually design such an algorithm for acyclic digraphs (as a pair of procedures illustrated by an example) just after Proposition 4.5.3. The criterion (iii) also provides the following characterization of line digraphs in terms of forbidden induced subdigraphs. Its proof is left as Exercise 4.5.

Corollary 4.5.2 *A directed pseudograph D is a line digraph if and only if D does not contain, as an induced subdigraph, any directed pseudograph that can be obtained from one of the directed pseudographs in Figure 4.5 (dotted arcs are missing) by adding zero or more arcs (other than the dotted ones).*

Observe that the digraph of order 4 in Figure 4.5 corresponds to the case of distinct vertices in Part (iii) of Theorem 4.5.1, and the two directed pseudographs of order 2 correspond to the cases $x = u \neq v = w$ and $u = w \neq v = x$, respectively.

Clearly, Theorem 4.5.1 implies a set of characterizations of the line digraphs of digraphs (without parallel arcs and loops). This can be found in [419]. Several characterizations of special classes of line digraphs and iterated line digraphs can be found in surveys by Hemminger and Beineke [419] and Prisner [614].

Many applications of line digraphs deal with the line digraphs of special families of digraphs, for example regular digraphs, in general, and complete

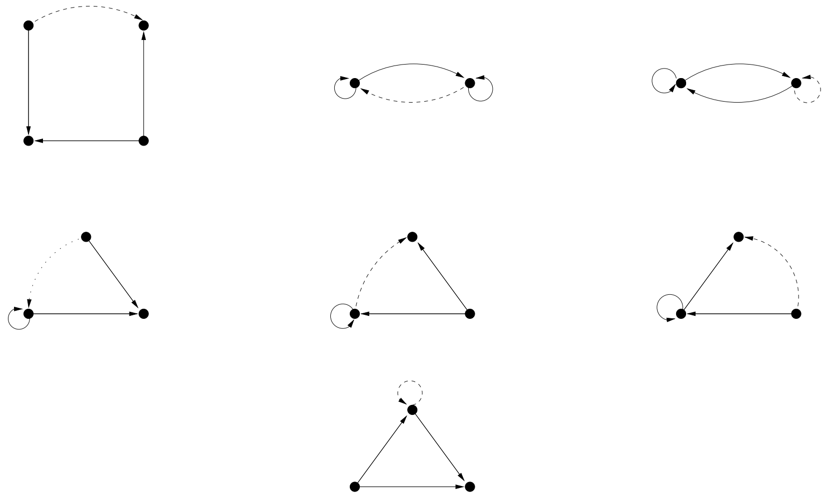

Figure 4.5 Forbidden directed pseudographs.

digraphs, in particular, see e.g., the papers [207] by Du, Lyuu and Hsu and [236] by Fiol, Yebra and Alegre. In Section 4.7, we need the following characterization, due to Harary and Norman, of the line digraphs of acyclic directed multigraphs. It is a specialization of Parts (i) and (ii) of Theorem 4.5.1. The proof is left as (an easy) Exercise 4.6.

Proposition 4.5.3 *[403] A digraph D is the line digraph of an acyclic directed multigraph if and only if D is acyclic and there exist two partitions $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ of $V(D)$ such that $A(D) = \cup_{i\in I} A_i \times B_i$.* □

We will now show how Proposition 4.5.3 can be used to recognize very effectively whether a given acyclic digraph R is the line digraph of another acyclic directed multigraph H, i.e., $R = L(H)$. The two procedures, which we construct and illustrate by Figure 4.8 can actually be used to recognize and represent (that is, to construct H such that $R = L(H)$) arbitrary line digraphs (see Theorem 4.5.1(i) and (ii)).

We first use Proposition 4.5.3 to check whether H above exists. The following procedure **Check-H** can be applied. Initially, all arcs and vertices of R are not marked. At every iteration, we choose an arc uv in R, which is not marked yet, and mark all vertices in $N^+(u)$ by 'B', all vertices in $N^-(v)$ by 'A' and all arcs in $(N^-(v), N^+(u))_R$ by 'C'. If $(N^-(v), N^+(u))_R \neq N^-(v) \times N^+(u)$ or if we mark a certain vertex or arc twice (starting from another arc $u'v'$) by the same symbol, then this procedure stops as there is no H such that $L(H) = R$. (We call these conditions **obstructions**.) If this procedure is performed to the end (i.e. every vertex and arc received a mark), then such H exists. It is not difficult to see, using Proposition 4.5.3, that Check-H correctly verifies whether H exists or not.

To illustrate Check-H, consider the digraph R_0 of Figure 4.8(a). Suppose that we choose the arc ab first. Then ab is marked, at the first iteration, together with the arcs af and ag. The vertex a receives 'A', the vertices b, f, g get 'B'. Suppose that fi is chosen at the second iteration. Then the arcs fh, fi, gh, gi are all marked at this iteration. The vertices f, g receive 'A', the vertices h, i 'B'. Suppose that bc is chosen at the third iteration. We see that this arc is the only arc marked at this iteration. The vertex b receives 'A', the vertex c 'B'. Finally, say, ce is chosen. Then both cd and ce are marked. The vertex c gets 'A', the vertices d, e receive 'B'. Thus, all arcs became marked with no obstruction happened. This means that there exists a digraph H_0 such that $H_0 = L(R_0)$.

Suppose now that H does exist. The following procedure **Build-H** constructs such a directed multigraph H. By Proposition 4.5.3, if H exists, then all arcs of R can be partitioned into arc sets of bipartite tournaments with partite sets A_i and B_i and arc sets $A_i \times B_i$. Let us denote these digraphs by $T_1, \dots, T_k$. (They can be computed by Check-H if we mark every $(N^-(v), N^+(u))_R$ not only by 'C' but also by a second mark 'i' starting from 1 and increasing by 1 at each iteration of the procedure.) We construct H as follows. The vertex set of H is $\{t_0, t_1, \dots, t_k, t_{k+1}\}$. The arcs of H are obtained by the following procedure. For each vertex v of R, we append one arc a_v to H according to the rules below:

(a) If $d_R(v) = 0$, then $a_v := (t_0, t_{k+1})$;
(b) If $d_R^+(v) > 0, d_R^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the index of T_i such that $v \in A_i$;
(c) If $d_R^+(v) = 0, d_R^-(v) > 0$, then $a_v := (t_j, t_{k+1})$, where j is the index of T_j such that $v \in B_j$;
(d) If $d_R^+(v) > 0, d_R^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in A_j \cap B_i$.

It is straightforward to verify that $R = L(H)$. Note that Build-H always constructs H with only one vertex of in-degree zero and only one vertex of out-degree zero.

To illustrate Build-H, consider R_0 of Figure 4.8 once again. Earlier we showed that there exists H_0 such that $R_0 = L(H_0)$. Now we will construct H_0. The previous procedure applied to verify the existence of H_0 has implicitly constructed the digraphs $T_1 = (\{a, b, f, g\}, \{ab, af, ag\})$, $T_2 = (\{f, g, h, i\}, \{fh, fi, gh, gi\})$, $T_3 = (\{b, c\}, \{bc\})$, $T_4 = (\{c, d, e\}, \{cd, ce\})$. Thus, H_0 has vertices $t_0, \dots, t_5$. Considering the vertices of R_0 in the lexicographic order, we obtain the following arcs of H_0 (in this order):

$$t_0t_1, t_1t_3, t_3t_4, t_4t_5, t_4t_5, t_1t_2, t_1t_2, t_2t_5, t_2t_5.$$

The directed multigraph H_0 is depicted in Figure 4.8(c). It is easy to check that $R_0 = L(H_0)$.

The iterated line digraphs are defined recursively: $L^1(D) = L(D)$, $L^{k+1}(D) = L(L^k(D))$, $k \geq 1$. It is not difficult to prove by induction (Exercise 4.8) that $L^k(D)$ is isomorphic to the digraph H, whose vertex set consists of walks of D of length k and a vertex $v_0v_1 \dots v_k$ (which is a walk in D) dominates the vertex $v_1v_2 \dots v_kv_{k+1}$ for every $v_{k+1} \in V(D)$ such that $v_kv_{k+1} \in A(D)$. New characterizations of line digraphs and iterated line digraphs are given by Liu and West [518].

The following proposition can be proved by induction on $k \geq 1$ (Exercise 4.10).

Proposition 4.5.4 *Let D be a strong d-regular digraph ($d > 1$) of order n and diameter t. Then $L^k(D)$ is of order d^kn and diameter $t + k$.* □

4.6 The de Bruijn and Kautz Digraphs and their Generalizations

The following problem is of importance in network design. Given positive integers n and d, construct a digraph D of order n and maximum out-degree at most d such that $\mathrm{diam}(D)$ is as small as possible and the vertex-strong connectivity $\kappa(D)$ is as large as possible. So we have a 2-objective optimization problem. For such a problem, in general, no solution can maximize/minimize both objective functions. However, for this specific problem, there are solutions, which (almost) maximize/minimize both objective functions. The aim of this section is to introduce these solutions, the de Bruijn and Kautz digraphs, as well as some of their generalizations. For more information on the above classes of digraphs, the reader may consult the survey [204] by Du, Cao and Hsu. For applications of these digraphs in design of parallel architectures and large packet radio networks, see e.g. the papers [113] by Bermond and Hell, [114] by Bermond and Peyrat and [649] by Samatan and Pradhan.

Let V be the set of vectors with t coordinates, $t \geq 2$, each taken from $\{0, 1, \dots, d-1\}$, $d \geq 2$. The **de Bruijn digraph** $D_B(d,t)$ is the directed pseudograph with vertex set V such that $(x_1, x_2, \dots, x_t)$ dominates $(y_1, y_2, \dots, y_t)$ if and only if $x_2 = y_1, x_3 = y_2, \dots, x_t = y_{t-1}$. See Figure 4.6 (a). Let $D_B(d,1)$ be the complete digraph of order d with loop at every vertex.

These directed pseudographs are named after de Bruijn who was the first to consider them in [185]. Clearly, $D_B(d,t)$ has d^t vertices and the out-pseudodegree and in-pseudodegree of every vertex of $D_B(d,t)$ equal d. This directed pseudograph has no parallel arcs and contains a loop at every vertex for which all coordinates are the same. It is natural to call $D_B(d,t)$

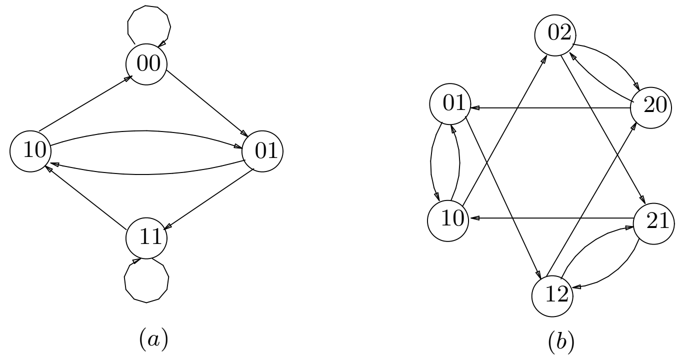

Figure 4.6 (a) The de Bruijn digraph $D_B(2,2)$; (b) The Kautz digraph $D_K(2,2)$.

***d*-pseudoregular** (recall that in the definition of semi-degrees we do not count loops).

Since $D_B(d,t)$ has loops at some vertices, the vertex-strong connectivity of $D_B(d,t)$ is at most $d-1$ (indeed, the loops can be deleted without the vertex-strong connectivity being changed). Imase, Soneoka and Okada [444] proved that $D_B(d,t)$ is $(d-1)$-strong, and moreover, for every pair $x \neq y$ of vertices there exist $d-1$ internally disjoint (x,y)-paths of length at most $t+1$. To prove this result we will use the following two lemmas. The proof of the first lemma, due to Fiol, Yebra and Alegre, is left as Exercise 4.11.

Lemma 4.6.1 *[236] For $t \geq 2$, $D_B(d,t)$ is the line digraph of $D_B(d,t-1)$.* □

Lemma 4.6.2 *Let x,y be distinct vertices of $D_B(d,t)$ such that $x \rightarrow y$. Then, there are $d-2$ internally disjoint (x,y)-paths different from xy, each of length at most $t+1$.*

Proof: Let $x = (x_1, x_2, \ldots, x_t)$ and $y = (x_2, \ldots, x_t, y_t)$. Consider the walk W_k given by $W_k = (x_1, x_2, \ldots, x_t), (x_2, \ldots, x_t, k), (x_3, \ldots, x_t, k, x_2), \ldots, (k, x_2, \ldots, x_t), (x_2, \ldots, x_t, y_t)$, where $k \neq x_1, y_t$. For each k, every internal vertex of W_k has coordinates forming the same multiset $M_k = \{x_2, \ldots, x_t, k\}$. Since for different k, the multisets M_k are different, the walks W_k are internally disjoint. Each of these walks is of length $t+1$. Therefore, by Proposition 1.4.1, $D_B(d,t)$ contains $d-2$ internally disjoint (x,y)-paths P_k with $A(P_k) \subseteq A(W_k)$. Since $k \neq x_1, y_t$, we may form the paths P_k such that none of them coincides with xy. □

Theorem 4.6.3 *[444] For every pair x,y of distinct vertices of $D_B(d,t)$, there exist $d-1$ internally disjoint (x,y)-paths, one of length at most t and the others of length at most $t+1$.*

Proof: By induction on $t \geq 1$. Clearly, the claim holds for $t = 1$ since $D_B(d, 1)$ contains, as spanning subdigraph, $\overleftrightarrow{K}_d$. For $t \geq 2$, by Lemma 4.6.1, we have that

$$D_B(d, t) = L(D_B(d, t-1)). \tag{4.5}$$

Let x, y be a pair of distinct vertices in $D_B(d, t)$ and let e_x, e_y be the arcs of $D_B(d, t-1)$ corresponding to vertices x, y due to (4.5). Let u be the head of e_x and let v be the tail of e_y.

If $u \neq v$, by the induction hypothesis, $D_B(d, t-1)$ has $d-1$ internally disjoint (u, v)-paths, one of length at most $t-1$ and the others of length at most t. The arcs of these paths together with arcs e_x and e_y correspond to $d-1$ internally disjoint (x, y)-paths in $D_B(d, t)$, one of length at most t and the others of length at most $t+1$.

If $u = v$, we have $x \rightarrow y$ in $D_B(d, t-1)$. It suffices to apply Lemma 4.6.2 to see that there are $d-1$ internally disjoint (x, y)-paths in $D_B(d, t)$, one of length one and the others of length at most $t+1$. □

By this theorem and Corollary 7.3.2, we conclude that $\kappa(D_B(d, t)) = d-1$. From Theorem 4.6.3 and Proposition 2.4.3, we obtain immediately the following simple, yet important property.

Proposition 4.6.4 *The de Bruijn digraph $D_B(d, t)$ achieves the minimum value t of diameter for directed pseudographs of order d^t and maximum out-degree at most d.* □

For $t \geq 2$, the **Kautz digraph** $D_K(d, t)$ is obtained from $D_B(d+1, t)$ by deletion of all vertices of the form $(x_1, x_2, \ldots, x_t)$ such that $x_i = x_{i+1}$ for some i. See Figure 4.6 (b). Define $D_K(d, 1) := \overleftrightarrow{K}_{d+1}$. Clearly, $D_K(d, t)$ has no loops and is a d-regular digraph. Since we have $d+1$ choices for the first coordinate of a vertex in $D_K(d, t)$ and d choices for each of the other coordinates, the order of $D_K(d, t)$ is $(d+1)d^{t-1} = d^t + d^{t-1}$. It is easy to see that Proposition 4.6.4 holds for the Kautz digraphs as well.

The following lemmas are analogous to Lemmas 4.6.1 and 4.6.2. Their proofs are left as Exercises 4.12 and 4.13.

Lemma 4.6.5 *For $t \geq 2$, the Kautz digraph $D_K(d, t)$ is the line digraph of $D_K(d, t-1)$.* □

Lemma 4.6.6 *Let xy be an arc in $D_K(d, t)$. There are $d-1$ internally disjoint (x, y)-paths different from xy, one of length at most $t+2$ and the others of length at most $t+1$.* □

The following result due to Du, Cao and Hsu [204] shows that the Kautz digraphs are better, in a sense, than de Bruijn digraphs from the local vertex-strong connectivity point of view. This theorem can be proved similarly to Theorem 4.6.3 and is left as Exercise 4.14.

Theorem 4.6.7 *[204] Let x, y be distinct vertices of $D_K(d,t)$. Then there are d internally disjoint (x,y)-paths in $D_K(d,t)$, one of length at most t, one of length at most $t+2$ and the others of length at most $t+1$.* $\square$

This theorem implies that $D_K(d,t)$ is d-strong.

The de Bruijn digraphs were generalized independently by Imase and Itoh [441] and Reddy, Pradhan and Kuhl [624] in the following way. We can transform every vector $(x_1, x_2, \ldots, x_t)$ with coordinates from $Z_d = \{0, 1, \ldots, d-1\}$ into an integer from $Z_{d^t} = \{0, 1, \ldots, d^t - 1\}$ using the polynomial $P(x_1, x_2, \ldots, x_t) = x_1 d^{t-1} + x_2 d^{t-2} + \ldots + x_t$. It is easy to see that this polynomial provides a bijection from Z_d^t to Z_{d^t}. Moreover, for $i, j \in Z_{d^t}$, $i \rightarrow j$ in $D_B(d,t)$ if and only if $j \equiv di + k \pmod{d^t}$ for some $k \in Z_d$.

Let d, n be two natural numbers such that $d < n$. The **generalized de Bruijn digraph** $D_G(d,n)$ is a directed pseudograph with vertex set Z_n and arc set

$$\{(i, di + k \pmod{n}\ ) : \ i, k \in Z_d\}.$$

For example, $V(D_G(2,5)) = \{0,1,2,3,4\}$ and $A(D_G(2,5)) = \{(0,0),(0,1),(1,2),(1,3),(2,4),(2,0),(3,1),(3,2),(4,3),(4,4)\}$.

Clearly, $D_G(d,n)$ is d-pseudoregular. It is not difficult to show that $\mathrm{diam}(D_G(d,n)) \leq \lceil \log_d n \rceil$. By Proposition 2.4.3, a digraph of maximum out-degree at most $d \geq 2$ and order n has a diameter at least $\lfloor \log_d n(d-1) + 1 \rfloor$. Thus, the generalized de Bruijn digraphs are of optimal or almost optimal diameter. It was proved, by Imase, Soneoka and Okada [443], that $D_G(d,n)$ is $(d-1)$-strong. It follows from these results that the generalized de Bruijn digraphs have almost minimum diameter and almost maximum vertex-strong connectivity.

The Kautz digraphs were generalized by Imase and Itoh [442]. Let n, d be two natural numbers such that $d < n$. The Imase-Itoh digraph $D_I(d,n)$ is the digraph with vertex set Z_n such that $i \rightarrow j$ if and only if $j \equiv -d(i+1)+k \pmod{n}$ for some $k \in Z_d$. It has been shown (for a brief account, see the paper [204]) by Du, Cao and Hsu, that $D_I(d,n)$ are of (almost) optimal diameter and vertex-strong connectivity.

Du, Hsu and Hwang [206] suggested a concept of digraphs extending both generalized the de Bruijn digraphs and the Imase-Ito digraphs. Let d, n be two natural numbers such that $d < n$. Given $q \in Z_n - \{0\}$ and $r \in Z_n$, **consecutive-d digraph** $D(d,n,q,r)$ is the directed pseudograph with vertex set Z_n such that $i \rightarrow j$ if and only if $j \equiv qi + r + k \pmod{n}$ for some $k \in Z_d$. Several results on diameter, vertex- and arc-strong connectivity and other properties of consecutive-d digraphs are given in [204]. In Section 5.11, we provide results on hamiltonicity of consecutive-d digraphs.

4.7 Series-Parallel Digraphs

In this section we study vertex series-parallel digraphs and arc series-parallel directed multigraphs. Vertex series-parallel digraphs were introduced by Lawler [510], and Monma and Sidney [568] as a model for scheduling problems. While vertex series-parallel digraphs continue to play an important role for the design of efficient algorithms in scheduling and sequencing problems, they have been extensively studied in their own right as well as in relations to other optimization problems (cf. the papers [36] by Baffi and Petreschi, [116] by Bertolazzi, Cohen, Di Battista, Tamassia and Tollis, [633] by Rendl and [682] by Steiner). Arc series-parallel directed multigraphs were introduced even earlier (than vertex series-parallel digraphs) by Duffin [209] as a mathematical model of electrical networks.

For an acyclic digraph D, let F_D (I_D) be the set of vertices of D of out-degree (in-degree) zero. To define vertex series-parallel digraphs, we first introduce **minimal vertex series-parallel (MVSP) digraphs** recursively.

The digraph of order one with no arc is an MVSP digraph. If $D = (V, A)$, $H = (U, B)$ is a pair of MVSP digraphs ($U \cap V = \emptyset$), so are the acyclic digraphs constructed by each of the following operations (see Figure 4.7):

(a) **Parallel composition:** $P = (V \cup U, A \cup B)$;
(b) **Series composition:** $S = (V \cup U, A \cup B \cup (F_D \times I_H))$.

It is interesting to note that we can embed every MVSP digraph D into the Cartesian plane such that if vertices u, v have coordinates (x_u, y_u) and (x_v, y_v), respectively, then there is a (u, v)-path in D if and only if $x_u \leq x_v$ and $y_u \leq y_v$. The proof of this non-difficult fact is given in the paper [726] by Valdes, Tarjan, and Lawler; see Exercise 4.15. See also Figure 4.9.

An acyclic digraph D is a **vertex series-parallel (VSP)** digraph if the transitive reduction of D is an MVSP digraph (see Subsection 4.3 for the definition of the transitive reduction). See Figure 4.8.

The following class of acyclic directed multigraphs, **arc series-parallel (ASP) directed multigraphs**, is related to VSP digraphs. The digraph $\vec{P}_2$ is an ASP directed multigraph. If D_1, D_2 is a pair of ASP directed multigraphs with $V(D_1) \cap V(D_2) = \emptyset$, then so are acyclic directed multigraphs constructed by each of the following operations (see Figure 4.10):

(a) **Two-terminal parallel composition:** Choose a vertex u_i of out-degree zero in D_i and a vertex v_i of in-degree zero in D_i for $i = 1, 2$. Identify u_1 with u_2 and v_1 with v_2;
(b) **Two-terminal series composition:** Choose $u \in F_{D_1}$ and $v \in I_{D_2}$ and identify u with v.

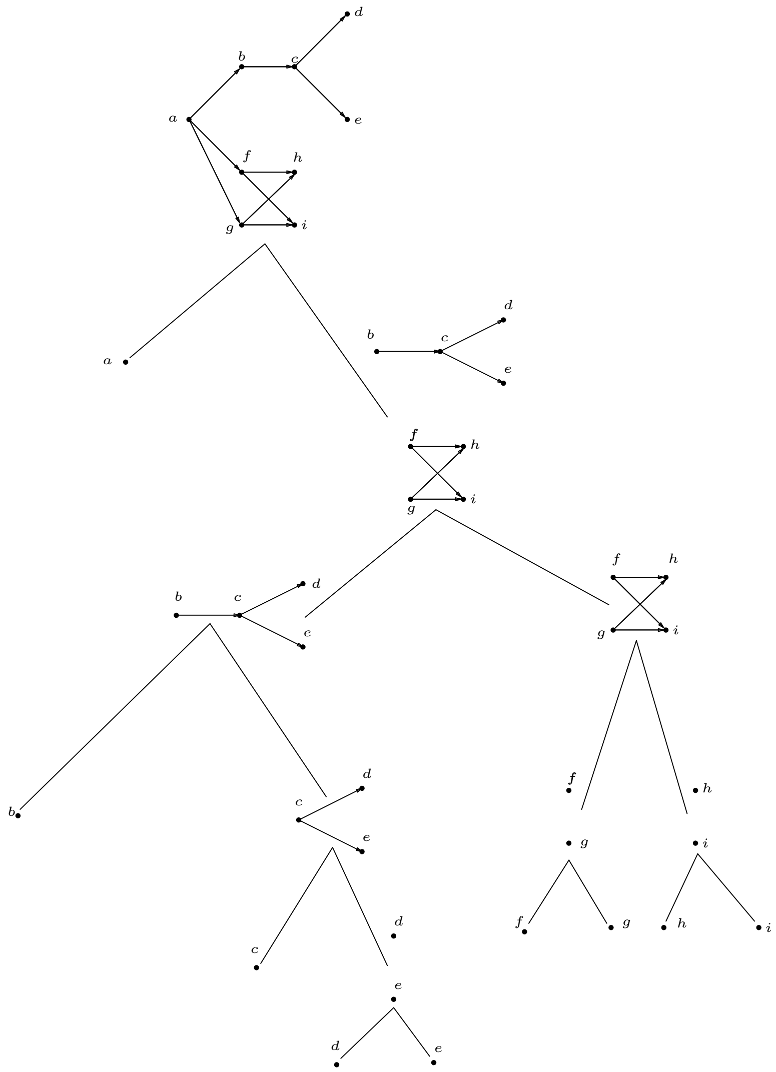

Figure 4.7 (De)construction of an MVSP digraph R_0 by series and parallel (de)compositions.

We refer the reader to the book [97] by Battista, Eades, Tamassia and Tollis for several algorithms for drawing graphs nicely, in particular drawing of ASP digraphs.

The next result shows a relation between the classes of digraphs introduced above.

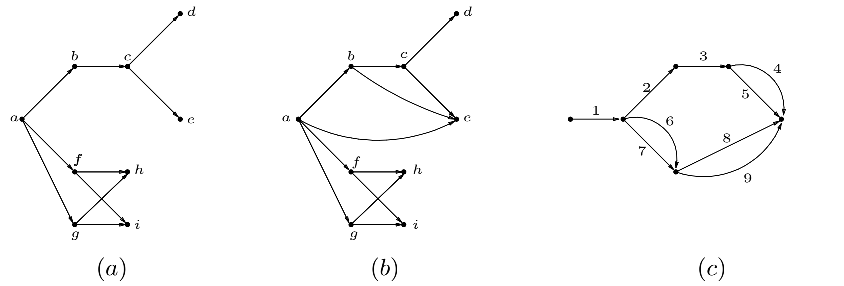

Figure 4.8 Series-parallel directed multigraphs: (a) an MVSP digraph R_0, (b) a VSP digraph R_1, (c) an AVSP directed multigraph H_0.

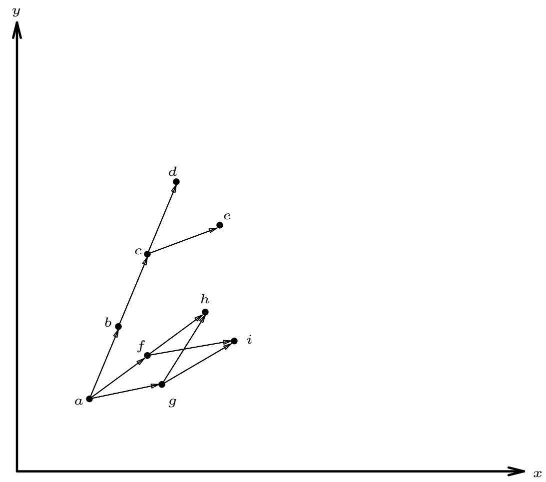

Figure 4.9 The MVSP digraph R_0 of Figure 4.7 embedded into the Cartesian plane such that for every (u, v)-path in R_0 we have $x_u \leq x_v$ and $y_u \leq y_v$ (and vice versa).

Theorem 4.7.1 *An acyclic directed multigraph D with a unique vertex of out-degree zero and a unique vertex of in-degree zero is ASP if and only if $L(D)$ is an MVSP digraph.*

Proof: This can be proved easily by induction on $|A(D)|$ using the following two facts:

(i) $L(\vec{P}_2) = \vec{P}_1$, which is an MVSP digraph;
(ii) The line digraph of the two-terminal series (parallel) composition of D_1 and D_2 is the series (parallel) composition of $L(D_1)$ and $L(D_2)$. □

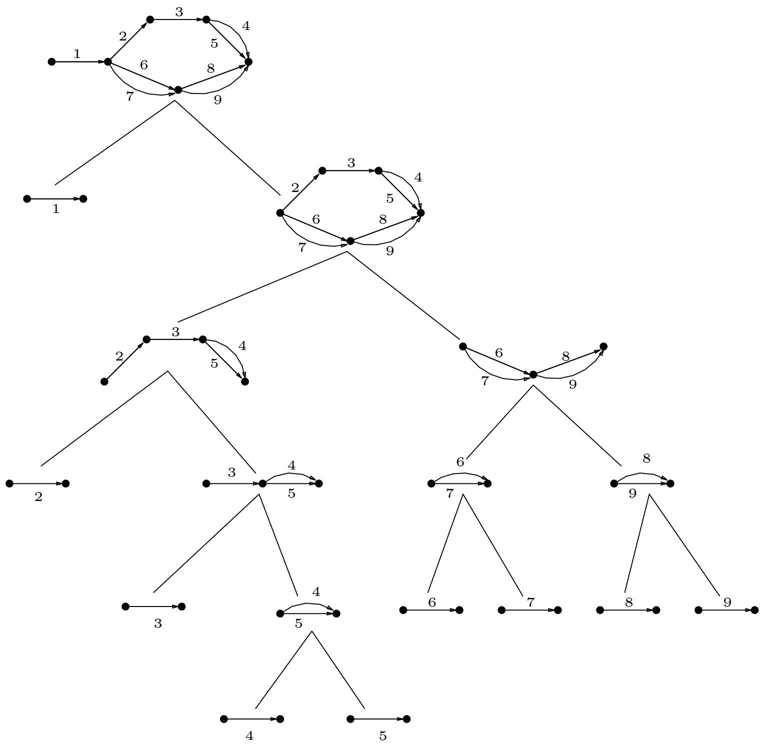

Figure 4.10 (De)construction of an ASP directed multigraph H_0 by two-terminal series and parallel (de)compositions.

It is easy to check that $L(H_0) = R_0$ for directed multigraphs H_0 and R_0 depicted in Figure 4.8. The following operations in a directed multigraph D are called **reductions:**

(a) **Series reduction:** Replace a path uvw, where $d_D^+(v) = d_D^-(v) = 1$ by the arc uw;
(b) **Parallel reduction:** Replace a pair of parallel arcs from u to v by just one arc from u to v.

The following proposition due to Duffin (see also the paper [726] by Valdes, Lawler and Tarjan) gives a characterization of ASP directed multigraphs. Its proof is left as Exercise 4.16.

Proposition 4.7.2 *[209] A directed multigraph is ASP if and only if it can be reduced to $\vec{P}_2$ by a sequence of series and parallel reductions.* □

The reader is advised to apply a sequence of series and parallel reductions to the directed multigraph H_0 of Figure 4.8 to obtain a digraph isomorphic to

$\vec{P}_2$. ¿From the algorithmic point of view, it is important that *every* sequence of series and parallel reductions transforms a directed multigraph to the same digraph. Indeed, this implies an obvious polynomial algorithm to verify if a given directed multigraph is ASP. The proof of the following result, due to Harary, Krarup and Schwenk, is left as Exercise 4.17.

Proposition 4.7.3 *[401] For every acyclic directed multigraph D, the result of application of series and parallel reductions until one can apply such reductions is a unique digraph H.* □

In [726], Valdes, Tarjan and Lawler showed how to construct a linear-time algorithm to recognize ASP directed multigraphs, which is based on Propositions 4.7.2 and 4.7.3. They also presented a more complicated linear-time algorithm to recognize VSP digraphs. Since we are limited in space, we will not discuss the details of the linear-time algorithms. Instead, we will consider the following simplified polynomial algorithm to recognize VSP digraphs.

VSP recognition algorithm:
Input: An acyclic digraph D.
Output: YES if D is VSP and NO, otherwise.

1. Compute the transitive reduction R of D.
2. Try to compute an acyclic directed multigraph H with $|I_H| = |F_H| = 1$ such that $L(H) = R$. If there is no such H, then output NO.
3. Verify whether H is an ASP directed multigraph. If it is so, then YES, otherwise, NO.

We prove first the correctness of this algorithm. If the output is YES, then, by Theorem 4.7.1, R is MVSP and thus D is VSP. If H is Step 2 is not found, then, by Theorem 4.7.1, R is not MVSP implying that D is not VSP. If H is not ASP, then R is not MVSP by the same theorem.

Now we prove that the algorithm is polynomial. Step 1 can be performed in polynomial time by Proposition 4.3.5. Step 2 can be implemented using Procedure Build-H described in the end of Section 4.5. This procedure implies that if there is an H such that $L(H) = R$, then there is such an H with additional property that $|I_H| = |F_H| = 1$. The procedure is polynomial. Finally, Step 3 is polynomial by the remark after Proposition 4.7.2.

4.8 Quasi-Transitive Digraphs

Quasi-transitive digraphs were introduced in Section 1.8. The aim of this section is to derive a recursive characterization of quasi-transitive digraphs which allows one to show that a number of problems for quasi-transitive digraphs including the longest path and cycle problems are polynomial time solvable (see Theorem 5.10.2). The characterization implies that every quasi-transitive digraph is totally Ψ-decomposable, where Ψ is the union of all transitive digraphs and all extended semicomplete digraphs. Our presentation is based on [79].

Proposition 4.8.1 *Let D be a quasi-transitive digraph. Suppose that $P = x_1x_2\dots x_k$ is a minimal (x_1,x_k)-path. Then the subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j{\rightarrow}x_i$ for every $2 \le i+1 < j \le k$, unless $k=4$, in which case the arc between x_1 and x_k may be absent.*

Proof: The cases $k = 2,3,4,5$ are easily verified. As an example, let us consider the case $k=5$. If x_i and x_j are adjacent and $2 \le i+1 < j \le 5$, then $x_j{\rightarrow}x_i$ since P is minimal. Since D is quasi-transitive, x_i and x_{i+2} are adjacent for $i = 1,2,3$. This and the minimality of P imply that $x_3{\rightarrow}x_1, x_4{\rightarrow}x_2$ and $x_5{\rightarrow}x_3$. From these arcs and the minimality of P we conclude that $x_5{\rightarrow}x_1$. Now the arcs x_4x_5 and x_5x_1 imply that $x_4{\rightarrow}x_1$. Similarly, $x_5{\rightarrow}x_1{\rightarrow}x_2$ implies $x_5{\rightarrow}x_2$.

The proof for the case $k \ge 6$ is by induction on k with the case $k=5$ as the basis. By induction, each of $D\langle\{x_1,x_2,\dots,x_{k-1}\}\rangle$ and $D\langle\{x_2,x_3,\dots,x_k\}\rangle$ is a semicomplete digraph and $x_j{\rightarrow}x_i$ for any $1 < j-i \le k-2$. Hence x_3 dominates x_1 and x_k dominates x_3 and the minimality of P implies that x_k dominates x_1. □

Corollary 4.8.2 *If a quasi-transitive digraph D has an (x,y)-path but x does not dominate y, then either $y{\rightarrow}x$, or there exist vertices $u,v \in V(D)-\{x,y\}$ such that $x{\rightarrow}u{\rightarrow}v{\rightarrow}y$ and $y{\rightarrow}u{\rightarrow}v{\rightarrow}x$.*

Proof: This is easy to deduce by considering a minimal (x,y)-path and applying Proposition 4.8.1. □

Lemma 4.8.3 *Suppose that A and B are distinct strong components of a quasi-transitive digraph D with at least one arc from A to B. Then $A{\mapsto}B$.*

Proof: Suppose A and B are distinct strong components such that there exists an arc from A to B. Then for every choice of $x \in A$ and $y \in B$ there exists a path from x to y in D. Since A and B are distinct strong components, none of the alternatives in Corollary 4.8.2 can hold and hence $x{\rightarrow}y$. □

Lemma 4.8.4 *[79] Let D be a strong quasi-transitive digraph on at least two vertices. Then the following holds:*

(a) $\overline{UG(D)}$ is disconnected;
(b) If S and S' are two subdigraphs of D such that $\overline{UG(S)}$ and $\overline{UG(S')}$ are distinct connected components of $\overline{UG(D)}$, then either $S{\mapsto}S'$ or $S'{\mapsto}S$, or both $S{\rightarrow}S'$ and $S'{\rightarrow}S$ in which case $|V(S)| = |V(S')| = 1$.

Proof: The statement (b) can be easily verified from the definition of a quasi-transitive digraph and the fact that S and S' are completely adjacent in D (Exercise 4.18). We prove (a) by induction on $|V(D)|$. Statement (a) is trivially true when $|V(D)| = 2$ or 3. Assume that it holds when $|V(D)| < n$ where $n > 3$.

Suppose that there is a vertex z such that $D - z$ is not strong. Then there is an arc from (to) every terminal (initial) component of $D - z$ to (from) z. Since D is quasi-transitive, the last fact and Lemma 4.8.3 imply that $X \rightarrow Y$ for every initial (terminal) strong component X (Y) of $D - z$. Similar arguments show that each strong component of $D - z$ either dominates some terminal component or is dominated by some initial component of $D - z$ (intermediate strong components satisfy both). These facts imply that z is adjacent to every vertex in $D - z$. Therefore, $\overline{UG(D)}$ contains a component consisting of the vertex z, implying that $\overline{UG(D)}$ is disconnected and (a) follows.

Assume that there is a vertex v such that $D - v$ is strong. Since D is strong, D contains an arc vw from v to $D - v$. By induction, $\overline{UG(D - v)}$ is not connected. Let connected components S and S' of $\overline{UG(D - v)}$ be chosen such that $w \in S$, $S \mapsto S'$ in D (here we use (b) and the fact that $D - v$ is strong). Then v is completely adjacent to S' in D (as $v \rightarrow w$). Hence $\overline{UG(S')}$ is a connected component of $\overline{UG(D)}$ and the proof is complete. □

The following theorem completely characterizes quasi-transitive digraphs in recursive sense (see also Figure 4.11).

Theorem 4.8.5 (Bang-Jensen and Huang) *[79] Let D be a digraph which is quasi-transitive.*

(a) If D is not strong, then there exist a transitive oriented graph T with vertices $\{u_1, u_2, \ldots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \ldots, H_t$ such that $D = T[H_1, H_2, \ldots, H_t]$, where H_i is substituted for u_i, $i = 1, 2, \ldots, t$.

(b) If D is strong, then there exists a strong semicomplete digraph S with vertices $\{v_1, v_2, \ldots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \ldots, Q_s]$, where Q_i is substituted for v_i, $i = 1, 2, \ldots, s$.

Proof: Suppose that D is not strong and let $H_1, H_2, \ldots, H_t$ be the strong components of D. According to Lemma 4.8.3, if there is an arc between H_i and H_j, then either $H_i \mapsto H_j$ or $H_j \mapsto H_i$. Now if $H_i \mapsto H_j \mapsto H_k$ then, by quasi-transitivity, $H_i \mapsto H_k$. So by contracting each H_i to a vertex h_i, we get a transitive oriented graph T with vertices $h_1, h_2, \ldots, h_t$. This shows that $D = T[H_1, H_2, \ldots, H_t]$.

Suppose now that D is strong. Let $Q_1, Q_2, \ldots, Q_s$ be the subdigraphs of D such that each $\overline{UG(Q_i)}$ is a connected component of $\overline{UG(D)}$. According to Lemma 4.8.4(a), each Q_i is either non-strong or just a single vertex. By

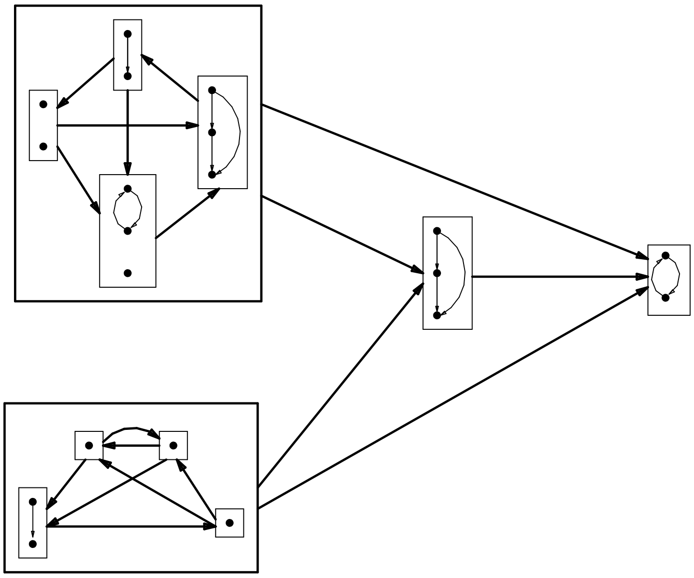

Figure 4.11 A decomposition of a non-strong quasi-transitive digraph. Big arcs between different boxed sets indicate that there is a complete domination in the direction shown.

Lemma 4.8.4(b) we obtain a strong semicomplete digraph S if each Q_i is contracted to a vertex. This shows that $D = S[Q_1, Q_2, \ldots, Q_s]$. $\square$

4.9 The Path-Merging Property and Path-Mergeable Digraphs

A digraph D is **path-mergeable**, if for any choice of vertices $x, y \in V(D)$ and any pair of internally disjoint (x, y)-paths P, Q, there exists an (x, y)-path R in D, such that $V(R) = V(P) \cup V(Q)$. We will see, in several places of this book, that the notion of a path-mergeable digraph is very useful for design of algorithms and proofs of theorems. This makes it worth while studying path-mergeable digraphs. The results presented in this section are adapted from [50], where the study of path-mergeable digraphs was initiated by Bang-Jensen.

We prove a characterization of path-mergeable digraphs, which implies that path-mergeable digraphs can be recognized efficiently.

Theorem 4.9.1 *A digraph D is path-mergeable if and only if for every pair of distinct vertices $x, y \in V(D)$ and every pair $P = xx_1 \ldots x_r y$, $P' = xy_1 \ldots y_s y$, $r, s \geq 1$ of internally disjoint (x, y)-paths in D, either there*

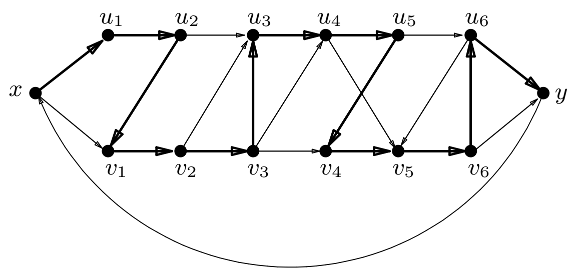

Figure 4.12 A digraph which is path-mergeable. The fat arcs indicate the path $xu_1u_2v_1v_2v_3u_3u_4u_5v_4v_5v_6u_6y$ from x to y which is obtained by merging the two (x, y)-paths $xu_1u_2u_3u_4u_5u_6y$ and $xv_1v_2v_3v_4v_5v_6y$.

exists an $i \in \{1, \ldots, r\}$*, such that* $x_i \to y_1$*, or there exists a* $j \in \{1, \ldots, s\}$*, such that* $y_j \to x_1$*.*

Proof: We prove 'only if' by induction on $r + s$. It is obvious for $r = s = 1$, so suppose that $r + s \geq 3$. If there is no arc between $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_s\}$, then clearly P, P' cannot be merged into one path. Hence we may assume without loss of generality that there is an arc x_iy_j for some i, j, $1 \leq i \leq r, 1 \leq j \leq s$. If $j = 1$ then the claim follows. Otherwise apply induction to the paths $P[x, x_i]y_j$, $xP'[y_1, y_j]$.

The proof of 'if' is left to the reader. It is similar to the proof of Proposition 4.9.3 below. □

The proof of the following result is left as Exercise 4.23.

Corollary 4.9.2 *Path-mergeable digraphs can be recognized in polynomial time.* □

The next result shows that, if a digraph is path-mergeable, then the merging of paths can always be done in a particularly nice way.

Proposition 4.9.3 *Let* D *be a digraph which is path-mergeable and let* $P = xx_1 \ldots x_ry$*,* $P' = xy_1 \ldots y_sy$*,* $r, s \geq 0$ *be internally disjoint* (x, y)*-paths in* D*. The paths* P *and* P' *can be merged into one* (x, y)*-path* P^* *such that vertices from* P *(respectively,* P'*) remain in the same order as on that path. Furthermore the merging can be done in at most* $2(r + s)$ *steps.*

Proof: We prove the result by induction on $r + s$. It is obvious if $r = 0$ or $s = 0$, so suppose that $r, s \geq 1$. By Theorem 4.9.1 there exists an i such that either $x_i \to y_1$ or $y_i \to x_1$. By scanning both paths forward one arc at a time, we can find i in at most $2i$ steps; suppose without loss of generality $x_i \to y_1$. By applying the induction hypothesis to the paths $P[x_i, x_r]y$ and $x_iP'[y_1, y_s]y$, we see that we can merge them into a single path Q in the required order-preserving way in at most $2(r + s - i)$ steps. The required path P^* is obtained by concatenating the paths $xP[x_1, x_i]$ and Q, and we have found it in at most $2(r+s)$ steps, as required. □

4.10 Locally In-Semicomplete and Locally Out-Semicomplete Digraphs

A digraph D is **locally in-semicomplete** (**locally out-semicomplete**) if, for every vertex x of D, the in-neighbours (out-neighbours) of x induce a semicomplete digraph. Clearly, the converse of a locally in-semicomplete digraph is a locally out-semicomplete digraph and vice versa. A digraph D is **locally semicomplete** if it is both locally in- and locally out-semicomplete. See Figure 4.13. Clearly every semicomplete digraph is locally semicomplete. A locally in-semicomplete digraph with no 2-cycle is a **locally in-tournament digraph**. Similarly, one can define **locally out-tournament digraphs** and **locally tournament digraphs**. For convenience, we will sometimes refer to locally tournament digraphs as **local tournaments** and to locally in-tournament (out-tournament) digraphs as **local in-tournaments** (**local out-tournaments**).

(a) (b)

Figure 4.13 (a) A locally out-semicomplete digraph which is not locally in-semicomplete; (b) A locally semicomplete digraph.

Proposition 4.10.1 by Bang-Jensen shows that locally in-semicomplete and locally out-semicomplete digraphs form subclasses of the class of path-mergeable digraphs. In particular, this means that every tournament is path-mergeable. In many theorems and algorithms on tournaments this property is of essential use. In some other cases, the very use of this property allows one to simplify proofs of results on tournaments and their generalizations or speed up algorithms on those digraphs.

Proposition 4.10.1 *[50] Every locally in-semicomplete (out-semicomplete) digraph is path-mergeable.*

Proof: Let D be a locally out-semicomplete digraph and let $P = y_1y_2\dots y_k$, $Q = z_1z_2\dots z_t$ be a pair of internally disjoint (x,y)-paths (i.e., $y_1 = z_1 = x$

and $y_k = z_t = y$). We show that there exists an (x, y)-path R in D, such that $V(R) = V(P) \cup V(Q)$. Our claim is trivially true when $|A(P)| + |A(Q)| = 3$. Assume now that $|A(P)| + |A(Q)| \geq 4$. Since D is out-semicomplete, either $y_2 \rightarrow z_2$ or $z_2 \rightarrow y_2$ (or both) and the claim follows from Theorem 4.9.1.

The proposition holds for locally in-semicomplete digraphs as they are the converses of locally out-semicomplete digraphs. □

The path-mergeability can be generalized in a natural way as follows. A digraph D is **in-path-mergeable** if, for every vertex $y \in V(D)$ and every pair P, Q of internally disjoint paths with common terminal vertex y, there is a path R such that $V(R) = V(P) \cup V(Q)$, the path R terminates at y and starts at a vertex which is the initial vertex of either P or Q (or, possibly, both). Observe that, in this definition, the initial vertices of paths P and Q may coincide. Therefore, every in-path-mergeable digraph is path-mergeable. However, it is easy to see that not every path-mergeable digraph is in-path-mergeable (see Exercise 4.19). A digraph D is **out-path-mergeable** if the converse of D is in-path-mergeable. Clearly, every in-path-mergeable (out-path-mergeable) digraph is locally in-semicomplete (locally out-semicomplete). The converse is also true (hence this is another way of characterizing locally in-semicomplete digraphs). The proof of Proposition 4.10.2 is left as Exercise 4.20.

Proposition 4.10.2 *Every locally in-semicomplete (out-semicomplete, respectively) digraph is in-path-mergeable (out-path-mergeable, respectively).*

□

Some simple, yet very useful, properties of locally in-semicomplete digraphs are described in the following results (in [81], by Bang-Jensen, Huang and Prisner, these results were proved for locally tournament digraphs only, so the statements below are their slight generalizations first stated by Bang-Jensen and Gutin [65]). Observe that a locally out-semicomplete digraph, being the converse of a locally in-semicomplete digraph, has similar properties (see Exercise 4.26). The claim of Theorem 4.10.4 is illustrated in Figure 4.14.

Lemma 4.10.3 *Every connected locally in-semicomplete digraph D has an out-branching.*

Proof: By Proposition 1.6.1, it suffices to prove that D has only one initial strong component. Assume that D has a pair D_1, D_2 of initial strong components (i.e. no arc enters D_1 or D_2). Let $y_i \in V(D_i)$, $i = 1, 2$, and let $P = x_1 x_2 \ldots x_s$ be a shortest path between $V(D_1)$ and $V(D_2)$ in the underlying graph G of D. Since no arc enters D_1 or D_2, there is an index $k \leq s$ such that $x_1 x_2 \ldots x_{k-1}$ is a path in D, but $x_k \rightarrow x_{k-1}$. Since D is in-semicomplete, the vertices x_{k-2} and x_k are adjacent. However, this contradicts the fact that P is a shortest path between $V(D_1)$ and $V(D_2)$ in G. □

Theorem 4.10.4 *Let D be a locally in-semicomplete digraph.*

(i) Let A and B be distinct strong components of D. If a vertex $a \in A$ dominates some vertex in B, then $a \mapsto B$.
(ii) If D is connected, then $SC(D)$ has an out-branching.

Proof: Let A and B be strong components of D for which there is an arc (a, b) from A to B. Since B is strong, there is a (b', b)-path in B for every $b' \in V(B)$. By the definition of locally in-semicomplete digraphs and the fact that there is no arc from B to A, we can conclude that $a \rightarrow b'$. This proves (i).

Part (ii) follows from the fact that $SC(D)$ is itself a locally in-tournament digraph and Lemma 4.10.3. □

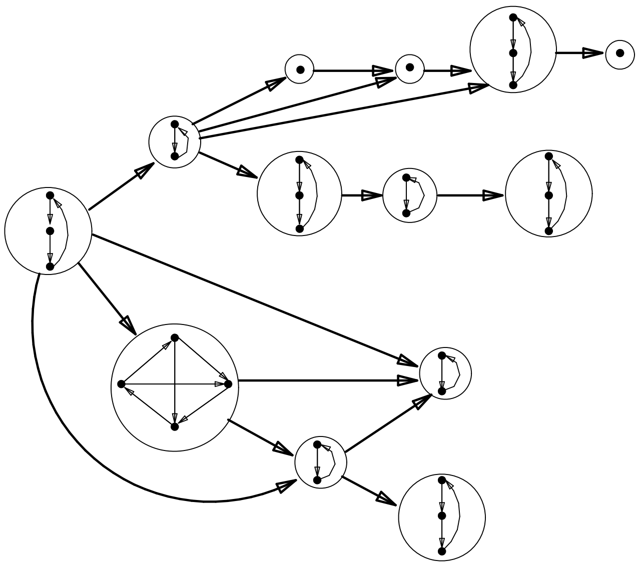

Figure 4.14 The strong decomposition of a non-strong locally in-semicomplete digraph. The big circles indicate strong components and a fat arc from a component A to a component B between two components indicates that there is at least one vertex $a \in A$ such that $a \mapsto B$.

4.11 Locally Semicomplete Digraphs

Locally semicomplete digraphs were introduced in 1990 by Bang-Jensen [44]. As shown in several places in our book, this class of digraphs has many nice properties in common with its proper subclass, semicomplete digraphs. The main aim of this section is to obtain a classification of locally semicomplete