Theorem 6.1.2 *[66] Let $D = (V, A)$ be a digraph which is either semicomplete bipartite or extended locally out-semicomplete and let $x \in V$. Then D has a hamiltonian path starting at x if and only if D contains a 1-path-cycle factor $\mathcal{F}$ of D such that the path of $\mathcal{F}$ starts at x, and, for every vertex y of $V - \{x\}$, there is an (x, y)-path*[1] *in D. Moreover, if D has a hamiltonian path starting at x, then, given a 1-path-cycle factor $\mathcal{F}$ of D such that the path of $\mathcal{F}$ starts at x, the desired hamiltonian path can be found in time $O(n^2)$.*

Proof: As the necessity is clear, we will only prove the sufficiency. Suppose that $\mathcal{F} = P \cup C_1 \cup ... \cup C_t$ is a 1-path-cycle factor of D that consists of a path P starting at x and cycles C_i, $i = 1, ..., t$. Suppose also that every vertex of D is reachable from x. Then, without loss of generality, there is a vertex of P that dominates a vertex of C_1. Let $P = x_1 x_2 \ldots x_p$, $C_1 = y_1 y_2 \ldots y_q y_1$, where $x = x_1$ and $x_k \rightarrow y_s$ for some $k \in \{1, 2, \ldots, p\}$, $s \in \{1, 2, \ldots, q\}$. We show how to find a new path starting at x which contains all the vertices of $V(P) \cup V(C_1)$. Repeating this process we obtain the desired path. Clearly, we may assume that $k < p$ and that x_p has no arc to $V(C_1)$.

Assume first that D is an extended locally out-semicomplete digraph. If P has a vertex x_i which is similar to a vertex y_j in C_1, then $x_i y_{j+1}, y_j x_{i+1} \in A$ and using these arcs we see that $P[x_1, x_i]C[y_{j+1}, y_j]P[x_{i+1}, x_p]$ is a path starting from x and containing all the vertices of $P \cup C_1$. If P has no vertex that is similar to a vertex in C_1, then we can apply the result of Exercise 4.37 to $P[x_k, x_p]$ and $x_k C_1[y_s, y_{s-1}]$ and merge these two paths into a path R starting from x_k and containing all the vertices of $P[x_k, x_p] \cup C_1$. Now, $P[x_1, x_{k-1}]R$ is a path starting at x and containing all the vertices of $P \cup C_1$.

Suppose now that D is semicomplete bipartite. Then either $y_{s-1} \rightarrow x_{k+1}$, which implies that $P[x_1, x_k]C_1[y_s, y_{s-1}]P[x_{k+1}, x_p]$ is a path starting at x and covering all the vertices of $P \cup C_1$, or $x_{k+1} \rightarrow y_{s-1}$. In the latter case, we consider the arc between x_{k+2} and y_{s-2}. If $y_{s-2} \rightarrow x_{k+2}$ we can construct the desired path, otherwise we continue to consider arcs between x_{k+3} and y_{s-3} and so on. If we do not construct the desired path in this way, then we find that the last vertex of P dominates a vertex in C_1, contradicting our assumption above.

Using the process above and breadth-first search, one can construct an $O(n^2)$-algorithm for finding the desired hamiltonian path starting at x. □

Just as the problem of finding a minimum path factor generalizes the hamiltonian path problem, we may generalize the problem of finding a hamiltonian path starting at a certain vertex to the problem of finding a path factor with as few paths as possible such that one of these paths starts at a specified vertex x. We say that a path factor **starts at *x*** if one of its paths starts at x and denote by $\mathrm{pc}_x(D)$ the minimum number of paths in a path factor that

[1] This is equivalent to saying that D has an out-branching with root x.

starts at x. The problem of finding a path factor with $\mathrm{pc}_x(D)$ paths which starts at x in a digraph D is called the **PFx** problem[2].

Let Φ_1 be the union of all semicomplete bipartite, extended locally semicomplete and acyclic digraphs. Using an approach similar to that taken in Section 5.10, Bang-Jensen and Gutin proved the following.

Theorem 6.1.3 *[66] Let D be a totally Φ_1-decomposable digraph. Then the PFx problem for D can be solved in time $O(|V(D)|^4)$.* □

6.2 Weakly Hamiltonian-Connected Digraphs

Recall that an $[x, y]$-path in a digraph $D = (V, A)$ is a path which either starts at x and ends at y or oppositely. We say that D is **weakly hamiltonian-connected** if it has a hamiltonian $[x, y]$-path (also called an **$[x, y]$-hamiltonian path**) for every choice of distinct vertices $x, y \in V$. Obviously deciding whether a digraph contains an $[x, y]$-hamiltonian path for some x, y is not easier than determining whether D has any hamiltonian path and hence for general digraphs this is an $\mathcal{NP}$-complete problem by Theorem 5.0.2 (see also Exercise 6.3). In this section we discuss various results that have been obtained for generalizations of tournaments. All of these results imply polynomial algorithms for finding the desired paths.

6.2.1 Results for Extended Tournaments

We start with a theorem due to Thomassen [698] which has been generalized to several classes of generalizations of tournaments as will be seen in the following subsections.

Theorem 6.2.1 *[698] Let $D = (V, A)$ be a tournament and let x_1, x_2 be distinct vertices of D. Then D has an $[x_1, x_2]$-hamiltonian path if and only if none of the following holds.*

(a) D is not strong and either none of x_1, x_2 belongs to the initial strong component of D or none of x_1, x_2 belongs to the terminal strong component (or both).
(b) D is strong and for $i = 1$ or 2, $D - x_i$ is not strong and x_{3-i} belongs to neither the initial nor the terminal strong component of $D - x_i$.
(c) D is isomorphic to one of the two tournaments in Figure 6.1 (possibly after interchanging the names of x_1 and x_2).

The following easy corollary is left as Exercise 6.4:

[2] Observe that $\mathrm{pc}_x(D) \leq \mathrm{pc}(D) + 1$ holds for every digraph D.

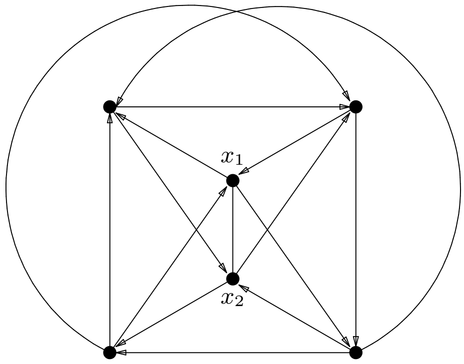

Figure 6.1 The exceptional tournaments in Theorem 6.2.1. The edge between x_1 and x_2 can be oriented arbitrarily.

Corollary 6.2.2 *[698] Let D be a strong tournament and let x, y, z be distinct vertices of D. Then D has a hamiltonian path connecting two of the vertices in the set $\{x, y, z\}$.* $\square$

Thomassen [698] used a nice trick in his proof of Theorem 6.2.1 by using Corollary 6.2.2 in the induction proof. We will give his proof below.

Proof of Theorem 6.2.1: Let x_1, x_2 be distinct vertices in a tournament D. It is easy to check that if any of (a)-(c) holds, then there is no $[x_1, x_2]$-hamiltonian path in D.

Suppose now that none of (a)-(c) hold. We prove by induction on n that D has an $[x_1, x_2]$-hamiltonian path. This is easy to show when $n \leq 4$ so assume now that $n \geq 5$ and consider the induction step with the obvious induction hypothesis. If D is not strong then let $D_1, D_2, \ldots, D_s$, $s \geq 2$ be the acyclic ordering of the strong components of D. Since (a) does not hold, we may assume without loss of generality that $x_1 \in V(D_1)$ and $x_2 \in V(D_s)$. Observe that D_1 has a hamiltonian path P_1 starting at x_1 (Exercise 6.1) and D_s has a hamiltonian path P_s ending at x_2. Let P_i be a hamiltonian path in D_i for each $i = 2, 3, \ldots, s-1$. Then $P_1 P_2 \ldots P_{s-1} P_s$ is an (x_1, x_2)-hamiltonian path.

If $D - x_i$ is not strong for $i = 1$ or 2, then we may assume without loss of generality that $i = 1$. Let $D'_1, \ldots, D'_p$, $p \geq 2$ be the acyclic ordering of the strong components of $D - x_1$. Since (b) does not hold we may assume, by considering the converse of D if necessary, that x_2 belongs to D'_p. Let y be any out-neighbour of x_1 in D'_1. Our argument for the previous case implies that there is a (y, x_2)-hamiltonian path P in $D - x_1$, implying that $x_1 P$ is an (x_1, x_2)-hamiltonian path in D. Hence we may assume that $D - x_i$ is strong for $i = 1, 2$.

If $D - \{x_1, x_2\}$ is not strong, then it is easy to prove that D has an (x_i, x_{3-i})-hamiltonian path for $i = 1, 2$ (Exercise 6.2). Hence we only need to consider the case when $D' = D - \{x_1, x_2\}$ is strong. Let $u_1u_2 \ldots u_{n-2}u_1$ be a hamiltonian cycle of D'. By considering the converse if necessary, we may assume that x_1 dominates u_1. Then D has an (x_1, x_2)-hamiltonian path unless x_2 dominates u_{n-2} so we may assume that is the case. By the same argument we see that either the desired path exists or x_1 dominates u_{n-3} and x_2 dominates u_{n-4}. Now it is easy to see that either the desired path exists, or $n-2$ is even and we have $x_1 \mapsto \{u_1, u_3, \ldots, u_{n-3}\}$, $x_2 \mapsto \{u_2, u_4, \ldots, u_{n-2}\}$. If x_1 or x_2 dominates any vertex other than those described above, then by repeating the argument above we see that either the desired path exists or $\{x_1, x_2\} \mapsto V(C)$, which is impossible since D is strong. Hence we may assume that

$$\begin{aligned} &\{u_2, u_4, \ldots, u_{n-2}\} \mapsto x_1 \mapsto \{u_1, u_3, \ldots, u_{n-3}\}, \\ &\{u_1, u_3, \ldots, u_{n-3}\} \mapsto x_2 \mapsto \{u_2, u_4, \ldots, u_{n-2}\} \end{aligned} \tag{6.1}$$

If $n = 6$, then using that (c) does not hold, it is easy to see that the desired path exists. So we may assume that $n \geq 8$. By induction, the theorem and hence also Corollary 6.2.2 holds for all tournaments on $n-2$ vertices. Thus D' has a hamiltonian path P which starts and ends in the set $\{u_1, u_3, u_5\}$ and by (6.1), P can be extended to an (x_1, x_2)-hamiltonian path of D. □

We now turn to extended tournaments. An extended tournament D does not always have a hamiltonian path, but, as we saw in Theorem 5.7.1, it does when the following obviously necessary condition is satisfied: there is a 1-path-cycle factor in D. Thus if we are looking for a sufficient condition for the existence of an $[x, y]$-hamiltonian path, we must require the existence on an $[x, y]$-path P such that $D - P$ has a cycle factor (this includes the case when P is already hamiltonian). Checking for such a path factor in an arbitrary digraph can be done in polynomial time using flows, see Exercise 3.62.

The next result is similar to the structure we found in the last part of the proof of Theorem 6.2.1.

Lemma 6.2.3 *[67] Suppose that D is a strong extended tournament containing two adjacent vertices x and y such that $D - \{x, y\}$ has a hamiltonian cycle C but D has no hamiltonian $[x, y]$-path. Then C is an even cycle, $N^+(x) \cap V(C) = N^-(y) \cap V(C)$, $N^-(x) \cap V(C) = N^+(y) \cap V(C)$, and the neighbours of x alternate between in-neighbours and out-neighbours around C.*

Proof: Exercise 6.5. □

Bang-Jensen, Gutin and Huang obtained the following characterization for the existence of an $[x, y]$-hamiltonian path in an extended tournament. Note the strong similarity with Theorem 6.2.1.

Theorem 6.2.4 *[67] Let D be an extended tournament and x_1, x_2 be distinct vertices of D. Then D has an $[x_1, x_2]$-hamiltonian path if and only if D has an $[x_1, x_2]$-path P such that $D - P$ has a cycle factor and D does not satisfy any of the conditions below:*

(a) D is not strong and either the initial or the terminal component of D (or both) contains none of x_1 and x_2;
(b) D is strong and the following holds for $i = 1$ or $i = 2$: $D - x_i$ is not strong and either x_{3-i} belongs to neither the initial nor the terminal component of $D - x_i$, or x_{3-i} does belong to the initial (terminal) component of $D - x_i$ but there is no (x_{3-i}, x_i)-path ((x_i, x_{3-i})-path) P' such that $D - P'$ has a cycle factor.
(c) $D, D - x_1$, and $D - x_2$ are all strong and D is isomorphic to one of the tournaments in Figure 6.1. □

The proof of this theorem in [67] is constructive and implies the following result (the proof is much more involved than that of Theorem 6.2.1). We point out that the proof in [67] makes explicit use of the fact that the digraphs have no 2-cycles. Hence the proof is only valid for extended tournaments and not for general extended semicomplete digraphs, for which the problem is still open.

Theorem 6.2.5 *[67] There exists an $O(\sqrt{n}m)$ algorithm to decide if a given extended tournament has a hamiltonian path connecting two specified vertices x and y. Furthermore, within the same time bound a hamiltonian $[x, y]$-path can be found if it exists.* □

Theorem 6.2.4 implies the following characterization of extended tournaments which are weakly hamiltonian-connected (see Exercise 6.7).

Theorem 6.2.6 *[67] Let D be an extended tournament. Then D is weakly hamiltonian-connected if and only if it satisfies each of the conditions below.*

(a) D is strongly connected.
(b) For every pair of distinct vertices x and y of D, there is an $[x, y]$-path P such that $D - P$ has a cycle factor.
(c) For each vertex x of D, $D - x$ has at most two strong components and if $D - x$ is not strong, then for each vertex y in the initial (respectively terminal) strong component, there is a (y, x)-path (respectively an (x, y)-path) P' such that $D - P'$ has a cycle factor.
(d) D is not isomorphic to any of the two tournaments in Figure 6.1. □

The following result generalizes Corollary 6.2.2. Note that we must assume the existence of the paths described below in order to have any chance of having a hamiltonian path with end vertices in the set $\{x, y, z\}$. The proof below illustrates how to argue with extended tournament structure.

Corollary 6.2.7 *[67] Let x, y and z be three vertices of a strong extended tournament D. Suppose that, for every choice of distinct vertices $u, v \in \{x, y, z\}$, there is a $[u, v]$-path P in D so that $D - P$ has a cycle factor. Then there is a hamiltonian path connecting two of the vertices in $\{x, y, z\}$.*

Proof: If both $D - x$ and $D - y$ are strong, then, by Theorem 6.2.4, either D has a hamiltonian path connecting x and y, or D is isomorphic to one of the tournaments in Figure 6.1, in which case there is a hamiltonian path connecting x and z. There is a similar argument if both $D - x$ and $D - z$, or $D - y$ and $D - z$ are strong. So, without loss of generality, assume that neither $D - x$ nor $D - y$ is strong. Let $S_1, S_2, \ldots, S_t$ be an acyclic ordering of the strong components of $D - x$. Note that S_t has an arc to x, since D is strong.

Suppose first that $y \in V(S_i)$ for some $1 < i < t$. We show that this implies that $D - y$ is strong, contradicting our assumption. Consider an $[x, y]$-path P and a cycle factor $\mathcal{F}$ of $D - P$. It is easy to see that P cannot contain any vertex of $S_{i+1}, \ldots, S_t$. Hence each of these strong components contains a cycle factor consisting of those cycles from $\mathcal{F}$ that are in S_j for $j = i{+}1, \ldots, t$. In particular (since it contains a cycle), each S_j has size at least 3 for $j = i+1, \ldots, t$. It also follows from the existence of P and $\mathcal{F}$ that every vertex in S_i is dominated by at least one vertex from $U = V(S_1) \cup \ldots \cup V(S_{i-1})$. Indeed, if some vertex $z \in V(S_i)$ is not dominated by any vertex from U, then using that $S_r \Rightarrow S_p$ for all $1 \leq r < p \leq t$ we get that z is similar to all vertices in U. However, this contradicts the existence of P and $\mathcal{F}$. Now it is easy to see that $D - y$ is strong since every vertex of $S_i - y$ is dominated by some vertex from $V(S_1) \cup \ldots \cup V(S_{i-1})$ and dominates a vertex in $V(S_{i+1}) \cup \ldots \cup V(S_t)$. Hence we may assume that y belongs to S_1 or S_t.

By considering the converse of D if necessary, we may assume that $y \in V(S_1)$. By Theorem 6.2.4(b) we may assume that there is no (y, x)-path W such that $D - W$ has a cycle factor. Thus it follows from the assumption of the corollary that there is an (x, y)-path $P' = v_1 v_2 \ldots v_r$, $v_1 = x, v_r = y$ such that $D - P'$ has a cycle factor $\mathcal{F}'$. Since $P' - x$ is contained in S_1, we can argue as above that each S_i, $i > 1$, has a cycle factor (inherited from $\mathcal{F}'$) and hence each S_i contains a hamiltonian cycle C_i, by Theorem 5.7.7.

Note that every vertex of S_1 which is not on P' belongs to some cycle of $\mathcal{F}'$ that lies entirely inside S_1. Hence, if $r = 2$ (that is, P' is just the arc $x \rightarrow y$), then it follows from Proposition 6.1.1 (which is also valid when the path in question has length zero) that S_1 contains a hamiltonian path starting at y. This path can easily be extended to a (y, x)-hamiltonian path in D, since each S_i, $i > 1$, is hamiltonian. Thus we may assume that $r \geq 3$.

If $S_1 - y$ is strong then $D - y$ is strong, contradicting our assumption above. Let $T_1, T_2, \ldots, T_s$, $s \geq 2$, be an acyclic ordering of the strong components of $S_1 - y$. Note that each $V(T_i)$ is either covered by some cycles from the cycle factor $\mathcal{F}'$ of $D - P'$ and hence T_i has a hamiltonian cycle (by Theorem 5.7.5), or is covered by a subpath of $P'[v_2, v_{r-1}]$ and some cycles (possibly zero) from $\mathcal{F}'$ and hence T_i has a hamiltonian path (by Theorem 5.7.1). Note also that there is at least one arc from y to T_1 and at least one arc from T_s to y. If T_1 contains a portion of $P'[v_2, v_{r-1}]$, then it is clear that T_1 contains v_2. But then $D - y$ is strong since $x \rightarrow v_2$, contradicting our assumption. So T_1 contains no vertices of $P'[v_1, v_{r-1}]$ and hence, by the remark above, T_1 has a hamiltonian cycle to which there is at least one arc from y. Using the structure derived above, it is easy to show that D has a (y, x)-hamiltonian path (Exercise 6.6). □

It can be seen from the results above that, when we consider weak hamiltonian-connectedness, extended tournaments have a structure which is closely related to that of tournaments. To see that Theorem 6.2.4 does not extend to general multipartite tournaments, consider the multipartite tournament D obtained from a hamiltonian bipartite tournament B with classes X and Y, by adding two new vertices x and y along with the following arcs: all arcs from x to X and from Y to x, all arcs from y to Y and X to y and an arc between x and y in any direction. It is easy to see that D satisfies none of the conditions (a)-(c) in Theorem 6.2.4, yet there can be no hamiltonian path with end vertices x and y in D because any such path would contain a hamiltonian path of B starting and ending in X or starting and ending in Y. Such a path cannot exist for parity reasons ($|X| = |Y|$). Note also that we can choose B so that the resulting multipartite tournament is highly connected.

Bang-Jensen and Manoussakis [86] characterized weakly hamiltonian-connected bipartite tournaments. In particular, they proved a necessary and sufficient condition for the existence of an $[x, y]$-hamiltonian path in a bipartite tournament. The statement of this characterization turns out to be quite similar to that of Theorem 6.2.4. The only difference between the statements of these two characterizations is in Condition (c): in the characterization for bipartite tournaments the set of forbidden digraphs is absolutely different and moreover infinite.

6.2.2 Results for Locally Semicomplete Digraphs

Our next goal is to describe the solution of the $[x, y]$-hamiltonian path problem for locally semicomplete digraphs. Notice that this solution also covers the case of semicomplete digraphs and so, in particular, it generalizes Theorem 6.2.1 to semicomplete digraphs.

We start by establishing notation for some special locally semicomplete digraphs. Up to isomorphism there is a unique strong tournament with four vertices. We denote this by T_4^1. It has the following vertices and arcs:

$$V(T_4^1) = \{a_1, a_2, a_3, a_4\}, A(T_4^1) = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_3, a_2a_4\}.$$

The semicomplete digraphs T_4^2, T_4^3, and T_4^4 are obtained from T_4^1 by adding some arcs, namely:

$$A(T_4^2) = A(T_4^1) \cup \{a_3a_1, a_4a_2\},$$

$$A(T_4^3) = A(T_4^1) \cup \{a_3a_1\}, A(T_4^4) = A(T_4^1) \cup \{a_1a_4\}.$$

Let $\mathcal{T}_4 = \{T_4^1, T_4^2, T_4^3, T_4^4\}$. It is easy to see that every digraph of $\mathcal{T}_4$ has a unique hamiltonian cycle and has no hamiltonian path between two vertices which are not consecutive on this hamiltonian cycle (such two vertices are called **opposite**).

Let $\mathcal{T}_6$ be the set of semicomplete digraphs with the vertex set $\{x_1, x_2, a_1, a_2, a_3, a_4\}$, each member D of $\mathcal{T}_6$ has a cycle $a_1a_2a_3a_4a_1$ and the digraph $D\langle\{a_1, a_2, a_3, a_4\}\rangle$ is isomorphic to one member of $\mathcal{T}_4$, in addition, $x_i \rightarrow \{a_1, a_3\} \rightarrow x_{3-i} \rightarrow \{a_2, a_4\} \rightarrow x_i$ for $i = 1$ or $i = 2$. It is straightforward to verify that $\mathcal{T}_6$ contains only two tournaments (denoted by T_6' and T_6''), namely the ones shown in Figure 6.1, and that $|\mathcal{T}_6| = 11$. Since none of the digraphs of $\mathcal{T}_4$ has a hamiltonian path connecting any two opposite vertices, no digraph of $\mathcal{T}_6$ has a hamiltonian path between x_1 and x_2.

For every even integer $m \geq 4$ there is only one 2-strong, 2-regular locally semicomplete digraph on m vertices, namely the second power $\vec{C}_m^2$ of an m-cycle (Exercise 6.8). We define

$$\mathcal{T}^* = \{\ \vec{C}_m^2 \mid m \text{ is even and } m \geq 4\}.$$

It is not difficult to prove that every digraph of $\mathcal{T}^*$ has a unique hamiltonian cycle and is not weakly hamiltonian-connected (Exercise 6.9, see also [47]). For instance, if the unique hamiltonian cycle of $\vec{C}_6^2$ is denoted by $u_1u_2u_3u_4u_5u_6u_1$, then $u_1u_3u_5u_1$ and $u_2u_4u_6u_2$ are two cycles of $\vec{C}_6^2$ and there is no hamiltonian path between any two vertices of $\{u_1, u_3, u_5\}$ or of $\{u_2, u_4, u_6\}$.

Let T_8^1 be the digraph consisting of $\vec{C}_6^2$ together with two new vertices x_1 and x_2 such that $x_1 \rightarrow \{u_1, u_3, u_5\} \rightarrow x_2 \rightarrow \{u_2, u_4, u_6\} \rightarrow x_1$. Furthermore, T_8^2 (T_8^3, respectively) is defined as the digraph obtained from T_8^1 by adding the arc x_1x_2 (the arcs x_1x_2 and x_2x_1, respectively). Let $\mathcal{T}_8 = \{T_8^1, T_8^2, T_8^3\}$. It is easy to see that every element of $\mathcal{T}_8$ is a 3-strong locally semicomplete digraph and has no hamiltonian path between x_1 and x_2.

Before we present the main result, we state the following two lemmas that were used in the proof of Theorem 6.2.10 by Bang-Jensen, Guo and Volkmann in [56]. The first lemma generalizes the structure found in the last part of the proof of Theorem 6.2.1.

Lemma 6.2.8 *[56] Let D be a strong locally semicomplete digraph on $n \geq 4$ vertices and x_1, x_2 two distinct vertices of D. If $D - \{x_1, x_2\}$ is strong, and $N^+(x_1) \cap N^+(x_2) \neq \emptyset$ or $N^-(x_1) \cap N^-(x_2) \neq \emptyset$, then D has a hamiltonian path connecting x_1 and x_2.*

Proof: Exercise 6.10. □

Another useful ingredient in the proof of Theorem 6.2.10 is the following linking result. An **odd chain** is the second power, $\vec{P}^2_{2k+1}$ for some $k \geq 1$, of a path on an odd number of vertices.

Lemma 6.2.9 *[56] Let D be a connected, locally semicompletedigraph with $p \geq 4$ strong components and acyclic ordering $D_1, D_2, ..., D_p$ of these. Suppose that $V(D_1) = \{u_1\}$ and $V(D_p) = \{v_1\}$ and that $D - x$ is connected for every vertex x. Then for every choice of $u_2 \in V(D_2)$ and $v_2 \in V(D_{p-1})$, D has two vertex disjoint paths P_1 from u_2 to v_1 and P_2 from u_1 to v_2 with $V(P_1) \cup V(P_2) = V(D)$ if and only if D is not an odd chain from u_1 to v_1.*

Proof: If D is an odd chain, it is easy to see that D has no two vertex disjoint (u_i, v_{3-i})-path for $i = 1, 2$ (Exercise 6.11). We prove by induction on p that the converse is true as well. Suppose that D is not an odd chain from u_1 to v_1. Since the subdigraph $D - x$ is connected for every vertex x, $|N^+(D_i)| \geq 2$ for all $i \leq p - 2$ and $|N^-(D_j)| \geq 2$ for all $j \geq 3$. If $p = 4$, then it is not difficult see that D has two vertex disjoint paths P_1 from u_2 to v_1 and P_2 from u_1 to v_2 with $V(P_1) \cup V(P_2) = V(D)$ (Exercise 6.13). If $p = 5$, it is also not difficult to check that D has the desired paths, unless D is a chain on five vertices. So we assume that $p \geq 6$. Now we consider the digraph D', which is obtained from D by deleting the vertex sets $\{u_1, v_1\}$, $V(D_2 - u_2)$ and $V(D_{p-1} - v_2)$.

Using the assumption on D, it is not difficult to show that D' is a connected, but not strongly connected locally semicompletedigraph with the acyclic ordering $\{u_2\}, D_3, D_4, \ldots, D_{p-2}, \{v_2\}$ of its strong components. Furthermore, for every vertex y of D', the subdigraph $D' - y$ is still connected. Let u be an arbitrary vertex of D_3 and v an arbitrary vertex of D_{p-2}. Note that there is a (u_1, u)-hamiltonian path P in $D\langle\{u_1, u\} \cup V(D_2 - u_2)\rangle$ and similarly there is a (v, v_1)-hamiltonian path Q in $D\langle\{v, v_1\} \cup V(D_{p-1} - v_2)\rangle$. Hence if D' has disjoint (u_2, v)-, (u, v_2)-paths which cover all vertices of D', then D has the desired paths. So we can assume D' has no such paths. By induction, D' is an odd chain from u_2 to v_2. Now using that D is not an odd chain from u_1 to v_1 it is easy to see that D has the desired paths. We leave the details to the reader. □

A weaker version of Lemma 6.2.9 was proved in [47, Theorem 4.5].

Below we give a characterization, due to Bang-Jensen, Guo and Volkmann for the existence of an $[x, y]$-hamiltonian path in a locally semicomplete digraph. Note again the similarity to Theorem 6.2.1.

Theorem 6.2.10 *[56] Let D be a connected locally semicomplete digraph on n vertices and x_1 and x_2 be two distinct vertices of D. Then D has no hamiltonian $[x_1, x_2]$-path if and only if one of the following conditions is satisfied:*

(1) D is not strong and either the initial or the terminal component of D (or both) contains none of x_1, x_2.
(2) D is strongly connected, but not 2-strong,
  (2.1) there is an $i \in \{1, 2\}$ such that $D - x_i$ is not strong and x_{3-i} belongs to neither the initial nor the terminal component of $D - x_i$;
  (2.2) $D - x_1$ and $D - x_2$ are strong, s is a separating vertex of D, $D_1, D_2, ..., D_p$ is the acyclic ordering of the strong components of $D - s$, $x_i \in V(D_\alpha)$ and $x_{3-i} \in V(D_\beta)$ with $\alpha \leq \beta - 2$. Furthermore, $V(D_{\alpha+1}) \cup V(D_{\alpha+2}) \cup ... \cup V(D_{\beta-1})$ contains a separating vertex of D, or $D' = D\langle V(D_\alpha) \cup V(D_{\alpha+1}) \cup ... \cup V(D_\beta)\rangle$ is an odd chain from x_i to x_{3-i} with $N^-(D_{\alpha+2}) \cap V(D - V(D')) = \emptyset$ and $N^+(D_{\beta-2}) \cap V(D - V(D')) = \emptyset$.
(3) D is 2-strong and is isomorphic to T_4^2 or to one member of $\mathcal{T}_6 \cup \mathcal{T}_8 \cup \mathcal{T}^*$ and x_1, x_2 are the corresponding vertices in the definitions. □

As an easy consequence of Theorem 6.2.10, we obtain a characterization of weakly hamiltonian-connected locally semicomplete digraphs. The proof is left to the interested reader as Exercise 6.12.

Theorem 6.2.11 *[56] A locally semicomplete digraph D with at least three vertices is weakly hamiltonian–connected if and only if it satisfies (a), (b) and (c) below:*

(a) D is strong,
(b) the subdigraph $D - x$ has at most two components for each vertex x of D,
(c) D is not isomorphic to any member of $\mathcal{T}_6 \cup \mathcal{T}_8 \cup \mathcal{T}^*$. □

6.3 Hamiltonian-Connected Digraphs

We now turn to hamiltonian paths with specified initial and terminal vertices. An **(x, y)-hamiltonian path** is a hamiltonian path from x to y. Clearly, asking for such a path in an arbitrary digraph is an even stronger requirement than asking for an $[x, y]$-hamiltonian path[3]. A digraph $D = (V, A)$ is **hamiltonian-connected** if D has an (x, y)-hamiltonian path for every choice of distinct vertices $x, y \in V$.

[3] We know of no class of digraphs for which the $[x, y]$-hamiltonian path problem is polynomially solvable, but the (x, y)-hamiltonian path problem is $\mathcal{NP}$-complete. For arbitrary digraphs they are equivalent from a complexity point of view (see Exercise 6.3).

No characterization for the existence of an (x, y)-hamiltonian path is known even for the case of tournaments[4]. Note however, that we sketch a polynomial algorithm for the problem in the next section, so in the algorithmic sense a good characterization does exist. The following very important partial result due to Thomassen will be used in the algorithm of the next section.

Theorem 6.3.1 (Thomassen) *[698] Let $D = (V, A)$ be a 2-strong semicomplete digraph with distinct vertices x, y. Then D contains an (x, y)-hamiltonian path if either (a) or (b) below is satisfied.*

(a) D contains three internally disjoint (x, y)-paths each of length at least two,
(b) D contains a vertex z which is dominated by every vertex of $V - x$ and D contains two internally disjoint (x, y)-paths each of length at least two.

□

In his proof Thomassen explicitly uses the fact that the digraph is allowed to have cycles of length 2. This simplifies the proof (which is still far from trivial), since one can use contraction to reduce to a smaller instance and then use induction.

An important ingredient in the proof of Theorem 6.3.1 as well as in several other proofs concerning the existence of an (x, y)-hamiltonian path in a semicomplete digraph D is to prove that D contains a spanning acyclic graph in which x can reach all other vertices and y can be reached by all other vertices. The reason for this can be seen from the following result which generalizes an observation by Thomassen in [698].

Proposition 6.3.2 *[50] Let D be a path-mergeable digraph. Then D has a hamiltonian (x, y)-path if and only if D contains a spanning acyclic digraph H in which $d^-_H(x) = d^+_H(y) = 0$ and such that, for every vertex $z \in V(D)$, H contains an (x, z)-path and a (z, y)-path.*

Proof: Exercise 6.15. □

Theorem 6.3.1 and Menger's theorem (see Theorem 7.3.1) immediately imply the following result. For another nice consequence see Exercise 6.16.

Theorem 6.3.3 *[698] If a semicomplete digraph D is 4-strong, then D is hamiltonian-connected.* □

Thomassen constructed an infinite family of 3-strongly connected tournaments with two vertices x, y for which there is no (x, y)-hamiltonian path [698]. Hence, from a connectivity point of view, Theorem 6.3.3 is the best possible.

[4] By this we mean a structural characterization involving only conditions that can be checked in polynomial time.

Theorem 6.3.3 is a very important result with several consequences. Thomassen has shown in several papers how to use Theorem 6.3.3 to obtain results on spanning collections of paths and cycles in semicomplete digraphs. See e.g. the papers [699, 701] by Thomassen and also Section 6.7. The following extension of Theorem 6.3.3 to extended tournaments has been conjectured by Bang-Jensen, Gutin and Huang:

Conjecture 6.3.4 *[67] If D is a 4-strong extended tournament with an (x, y)-path P such that $D - P$ has a cycle factor, then D has an (x, y)-hamiltonian path.*

Extending Theorem 6.3.3 to locally semicomplete digraphs, Guo [342] proved the following:

Theorem 6.3.5 (Guo) *[342] Let D be a 2-strong locally semicomplete digraph and let x, y be two distinct vertices of D. Then D contains a hamiltonian path from x to y if (a) or (b) below is satisfied.*

(a) There are three internally disjoint (x, y)-paths in D, each of which is of length at least 2 and D is not isomorphic to any of the digraphs T_8^1 and T_8^2 (see the definition in the preceding section).
(b) The digraph D has two internally disjoint (x, y)-paths P_1, P_2, each of which is of length at least 2 and a path P which either starts at x, or ends at y and has only x or y in common with P_1, P_2 such that $V(D) = V(P_1) \cup V(P_2) \cup V(P)$. Furthermore, for any vertex $z \notin V(P_1) \cup V(P_2)$, z has a neighbour on $P_1 - \{x, y\}$ if and only if it has a neighbour on $P_2 - \{x, y\}$. □

Since neither of the two exceptions in (a) is 4-strong, Theorem 6.3.5 implies the following:

Corollary 6.3.6 *[342] If a locally semicomplete digraph is 4-strong, then it is hamiltonian-connected.* □

In [341] Guo used Theorem 6.3.5 to give a complete characterization of those 3-strongly connected arc-3-cyclic (that is, every arc is in a 3-cycle) locally tournament digraphs with no hamiltonian path from x to y for specified vertices x and y. In particular this characterization shows that there exist infinitely many 3-strongly connected digraphs which are locally tournament digraphs (but not semicomplete digraphs) and are not hamiltonian-connected. Thus, as far as this problem is concerned, it is not only the subclass of semicomplete digraphs which contain difficult instances within the class of locally semicomplete digraphs. It should be noted that Guo's proof does not rely on Theorem 6.3.3. However, due to the non-semicomplete exceptions mentioned above, it seems unlikely that a much simpler proof of Corollary 6.3.6 can be found using Theorem 6.3.3 and Theorem 4.11.15.

Not surprisingly, there are also several results, such as the following by Lewin, on hamiltonian-connectivity in digraphs with many arcs.

Theorem 6.3.7 *[514] Every digraph on $n \geq 3$ vertices and at least $(n-1)^2+1$ arcs is hamiltonian-connected.* □

If a digraph D is hamiltonian-connected, then D is also hamiltonian (since every arc is in a hamiltonian cycle). The next result, due to Bermond, shows that we only need a slight strengthening of the degree condition in Theorem 5.6.3 to get a sufficient condition for strong hamiltonian-connectivity.

Theorem 6.3.8 *[108] Every digraph D on n vertices which satisfies $\delta^0(D) \geq \frac{n+1}{2}$ is hamiltonian-connected.* □

If we just ask for weak hamiltonian-connectness then Overbeck-Larisch showed that we can replace the condition on the semi-degrees by a condition on the degrees:

Theorem 6.3.9 *[597] Every 2-strong digraph on n vertices and minimum degree at least $n+1$ is weakly hamiltonian-connected.* □

Thomassen asked whether all 3-strong digraphs $D=(V,A)$ on n vertices with $d^+(x)+d^-(x) \geq n+1$ for all $x \in V$ are necessarily hamiltonian-connected. However, this is not the case, as was shown by Darbinyan [179].

6.4 Finding a Hamiltonian (x, y)-Path in a Semicomplete Digraph

In this section we discuss algorithmic aspects of the (x, y)-hamiltonian path problem for semicomplete digraphs. The main result is the following by Bang-Jensen, Manoussakis and Thomassen:

Theorem 6.4.1 *[87] The (x, y)-hamiltonian path problem is polynomially solvable for semicomplete digraphs.* □

We will not give the proof of this difficult result here, but rather outline the most interesting ingredients in the non-trivial proof in [87]. As usual, we will always use n to denote the number of vertices of the digraph in question.

The first lemma is quite simple to prove, but it turns out to be very useful for the design of the algorithm of Theorem 6.4.1.

If x, w, z are distinct vertices of a digraph D, then we use the notation $Q_{x,z}$, $Q_{.,w}$ to denote two disjoint paths such that the first path is an (x, z)-path, the second path has terminal vertex w, and $V(Q_{x,z}) \cup V(Q_{.,w}) = V(D)$. Similarly $Q_{z,x}$ and $Q_{w,.}$ denote two disjoint paths, such that the first path is a (z, x)-path, the second path has initial vertex w, and $V(Q_{z,x}) \cup V(Q_{w,.}) = V(D)$.

Lemma 6.4.2 *[87] Let x, w, z be distinct vertices in a semicomplete digraph T, such that there exist internally disjoint (x, w)-, (x, z)-paths P_1, P_2 in T. Let $R = T - V(P_1) \cup V(P_2)$.*

(a) There are either $Q_{x,w}$, $Q_{\cdot,z}$ or $Q_{x,z}$, $Q_{\cdot,w}$ in T, unless there is no arc from R_t to $V(P_1) \cup V(P_2) - x$, where R_t is the terminal component of $T\langle R \rangle$.
(b) In the case when there is an arc from R_t to $V(P_1) \cup V(P_2) - x$ we can find one of the pairs of paths, such that the path with only one end vertex specified has length at least one, unless $V(P_1) \cup V(P_2) = \{w, x, z\}$.
(c) Moreover there is an $O(n^2)$ algorithm to find one of the pairs of paths above if they exist.

Proof: If $R = \emptyset$ then both pairs of paths exist. Hence we may assume that $R \neq \emptyset$. Assume there is an arc uv where $u \in R_t$ and $v \in (V(P_1) \cup V(P_2)) - x$. Assume without loss of generality that $v \in P_1$. Since $u \in R_t$, $T\langle R \rangle$ has a hamiltonian path Q ending at u and starting at some vertex y. By Proposition 4.10.2, the semicomplete digraph $T\langle R \cup V(P_1) - x \rangle$ has a hamiltonian path starting either at y or the successor of x on P_1 and ending in w. This path together with P_2 forms the desired pair of paths $Q_{x,z}, Q_{\cdot,w}$. This proves (a). It is easy to verify (b) by the same argument. As the strong components of $T\langle R \rangle$ and a hamiltonian cycle in each of them can be found in $O(n^2)$ time (Theorem 5.5.2), we can find Q and $Q_{x,z}, Q_{\cdot,w}$ in $O(n^2)$ time. □

We point out that the proof above shows that Lemma 6.4.2 is valid also for digraphs that are locally in-semicomplete.

The following lemma allows one to use symmetry and thereby reduces the number of cases to consider when looking for an (x, y)-hamiltonian path.

Lemma 6.4.3 *Let T be a semicomplete digraph and x, y vertices of T, such that there exist 2 internally disjoint (x, y)-paths and an (x, y)-separator $\{u, v\}$ in T. Suppose that u, v do not induce a 2-cycle, say, $v \not\rightarrow u$. Let T' denote the semicomplete digraph obtained from T, by adding the arc $v \rightarrow u$. Then T has an (x, y)-hamiltonian path if and only if T' has an (x, y)-hamiltonian path.*

Proof: Exercise 6.18. □

The next result shows that either T is 2-strong or we can reduce the problem to smaller instances.

Lemma 6.4.4 *[87] If T is not 2-strong then either the desired path exists in T, or we can reduce the problem to one or two smaller problems, such that in the latter case the total size of the subproblems is at most $n + 1$.* □

We now outline the major steps of the algorithm in [87] for the (x, y)-hamiltonian path problem. First we make some assumptions which do not change the problem.

We assume that there is no arc from x to y and that neither x nor y are contained in a 2-cycle (if there is such a cycle containing x (y), then delete the arc entering x (leaving y)). It is easy to see that the new semicomplete digraph has an (x, y)-hamiltonian path if and only if the original digraph has one. So we assume that the input is a semicomplete digraph T which has the form above. In order to refer to smaller versions of the same problem we refer to the problem as the **hamiltonian problem**. Note that by Lemma 6.4.4 we may assume that T is 2-strong (otherwise we just consider smaller subproblems).

With the assumptions above it follows from Theorem 6.3.1 that, if there are three internally disjoint (x, y)-paths in T, then the desired hamiltonian path exists. Thus, by Lemma 6.4.4, the interesting part is when T is 2-strong and there are two but not three internally disjoint (x, y)-paths. By Menger's theorem (which we study in Chapter 7) we may thus assume that there exists an (x, y)-separator of size two in T.

The next theorem by Bang-Jensen, Manoussakis and Thomassen generalizes Theorem 6.3.1. It is very important for the proof of Theorem 6.4.1, because it corresponds to a case when no reduction is possible (see the description of the algorithm below) and hence one has to prove the existence of the desired path directly. Recall that for specified distinct vertices s, t, an (s, t)-separator is a subset $S \subseteq V - \{s, t\}$ such that $D - S$ has no (s, t)-path. An (s, t)-separator is **trivial** if either s has out-degree zero or t has in-degree zero in $D - S$.

Theorem 6.4.5 *[87] Let T be a 2-strong semicomplete digraph on at least 10 vertices and let x, y be vertices of T such that $y \mapsto x$. Suppose that $T - x, T - y$ are both 2-strong. If all (x, y)-separators consisting of two vertices (if any exist) are trivial, then T has an (x, y)-hamiltonian path.* $\square$

Besides the results mentioned above the algorithm uses the following results:

Lemma 6.4.6 *[87] Suppose T is 2-strong and there exists a non-trivial separator $\{u, v\}$ of x, y. Let A, B denote a partition of $T - \{u, v\}$ such that $y \in A, x \in B$ and $A \mapsto B$. Let $T' = T\langle A \cup \{u, v\}\rangle, T'' = T\langle B \cup \{u, v\}\rangle$. We can reduce the hamiltonian problem to at most four hamiltonian problems such that one has size* $\max\{|A|, |B|\} + 2$ *or* $\max\{|A|, |B|\} + 3$ *and the others (if any) have size at most* $\min\{|A|, |B|\} + 3$. $\square$

Lemma 6.4.7 *[87] Suppose that T is 2-strong, $n \geq 6$, and all (x, y)-separators of size 2 x, y are trivial. If $T - x$ or $T - y$ is not 2-strong, then either the desired path exists in T, or we can reduce the problem to one or two smaller problems, such that in the latter case, the total size of the subproblems is at most $n + 2$.* $\square$

The hamiltonian algorithm

1. If $n \leq 9$, then settle the problem in constant time.
2. If T is not 2-strong, then using Lemma 6.4.4 we settle the problem, or reduce to smaller instances of the hamiltonian problem.

3. If there are no (x, y)-separators of size 2, then T has the desired path, by Theorem 6.3.1.
4. If all (x, y)-separators of size 2 are trivial, we check if $T - x$ and $T - y$ are 2-strong. Then we settle or reduce the problem using Theorem 6.4.5 or Lemma 6.4.7.
5. Let $\{u, v\}$ be a non-trivial (x, y)-separator and let A, B form a partition of $T - \{u, v\}$, such that $y \in A, x \in B$ and $A \mapsto B$. (Such a partition can be found in time $O(n^2)$, by letting B be the vertices which in $T - \{u, v\}$ can be reached from x by a directed path and then taking $A = V - B - \{u, v\}$.) Also, if necessary, add an arc to make u, v induce a 2-cycle. This does not change the problem, by Lemma 6.4.3.
6. Use the algorithmic version of Lemma 6.4.2 to find $Q_{x,u}$, $Q_{.,v}$ or $Q_{x,v}$, $Q_{.,u}$ in $T'' = T(B \cup \{u, v\})$, and use an analogous algorithm to find $Q_{u,y}$, $Q_{v,.}$ or $Q_{v,y}$, $Q_{u,.}$ in $T' = T(A \cup \{u, v\})$. These paths exist, since T is 2-strong, and the paths with one end vertex unspecified can be chosen of length at least one, since A, B both have size at least 2 (here we used that $\{u, v\}$ is a non-trivial separator).
7. If these paths match then T has the desired (x, y)-hamiltonian path. So suppose (by renaming u, v if necessary) that we find $Q_{x,u}$, $Q_{.,v}$ in T'' and $Q_{u,y}$, $Q_{v,.}$ in T'.
8. Using Lemma 6.4.6 we can now reduce the problem to smaller instances of the hamiltonian problem.

In Step 7 we say that the two sets of paths in T'' and T' **match** if the following holds: the paths are P_1 from x to w and P_2 from p to z in T'' and R_1 from r to y and R_2 from s to q in T' where $\{w, z\} = \{r, s\} = \{u, v\}$ and $w = s$ and $z = r$. In this case the path $P_1 R_2 P_2 R_1$ is the desired hamiltonian path since $q \to p$ by the definition of B in Step 5.

The complexity of the algorithm outlined above is $O(n^5)$ (in fact, it is $O(n^{4+\epsilon})$ for every $\epsilon > 0$). No attempt was made in [87] to improve the complexity, but it seems quite difficult to improve it very much.

It is interesting to note that the algorithm described above cannot be easily modified to solve the problem of finding the longest path with specified initial and terminal vertex in a semicomplete digraph. In several places we explicitly use that we are searching for a hamiltonian path. There also does not seem to be any simple reduction of this problem to the problem of deciding the existence of a hamiltonian path from x to y.

Conjecture 6.4.8 *[65] There exists a polynomial algorithm which, given a semicomplete digraph D and two distinct vertices x and y of D, finds a longest (x, y)-path.*

Note that, if we ask for the longest $[x, y]$-path in a tournament, then this can be answered using Theorem 6.2.1 (see Exercise 6.19).

Conjecture 6.4.9 *[65] There exists a polynomial algorithm which, given a digraph D that is either extended semicomplete or locally semicomplete, and two distinct vertices x and y of D, decides whether D has an (x,y)-hamiltonian path and finds such a path if one exists.*

6.5 Pancyclicity of Digraphs

A digraph D of order n is **pancyclic** if it has cycles of all lengths $3,4,\dots,n$. We say that D is **vertex-pancyclic** if for any $v \in V(D)$ and any $k \in \{3,4,...,n\}$ there is a cycle of length k containing v. We also say that D is **(vertex-)m-pancyclic** if D contains a k-cycle (every vertex of D is on a k-cycle) for each $k = m, m+1, \dots, n$. Note that some early papers on pancyclicity in digraphs require that D is (vertex-)2-pancyclic in order to be (vertex-)pancyclic (see e.g. the survey [115] by Bermond and Thomassen). We feel that this definition is too restrictive, since often one can prove pancyclicity results for much broader classes of digraphs when the 2-cycle is omitted from the requirement.

6.5.1 (Vertex-)Pancyclicity in Degree-Constrained Digraphs

The following claim is due to Alon and Gutin:

Lemma 6.5.1 *[11] Every directed graph $D=(V,A)$ on n vertices for which $\delta^0(D) \geq n/2+1$ is vertex-2-pancyclic.*

Proof: Let $v \in V$ be arbitrary. By Corollary 5.6.3 there is a Hamilton cycle $u_1u_2\dots u_{n-1}u_1$ in $D-v$. If there is no cycle of length k through v then for every i, $|N^+(v) \cap \{u_i\}| + |N^-(v) \cap \{u_{i+k-2}\}| \leq 1$, where the indices are computed modulo $n-1$. By summing over all values of i, $1 \leq i \leq n-1$, we conclude that $|N^-(v)| + |N^+(v)| \leq n-1$, contradicting the assumption that all in-degrees and out-degrees exceed $n/2$. □

Thomassen [696] proved that just by adding one to the degree condition for hamiltonicity in Theorem 5.6.7 one obtains cycles of all possible lengths in the digraphs satisfying the degree condition.

Theorem 6.5.2 *[696] Let D be a strong digraph on n vertices such that $d(x)+d(y) \geq 2n$ whenever x and y are nonadjacent. Then either D has cycles of all lengths $2,3,\dots,n$, or D is a tournament (in which case it has cycles of all lengths $3,4,\dots,n$) or n is even and D is isomorphic to $\overleftrightarrow{K}_{\frac{n}{2},\frac{n}{2}}$.* □

The following example from [696] shows that $2n$ cannot be replaced by $2n-1$ in Theorem 6.5.2. For some $m \leq n$ let $D_{n,m}=(V,A)$ be the digraph with vertices $V = \{v_1, v_2, \ldots, v_n\}$ and arcs $A = \{v_iv_j | i < j \text{ or } i = j+1\} - \{v_iv_{i+m-1} | 1 \leq i \leq n-m+1\}$. We leave it as Exercise 6.20 to show that $D_{n,m}$ is strong, has no m-cycle and if $m > (n+1)/2$, then $D_{n,m}$ satisfies Meyniel's condition for hamiltonicity (Theorem 5.6.7). In [176] Darbinyan characterizes those digraphs which satisfy Meyniel's condition, but are not pancyclic.

Theorem 6.5.2 extends Moon's theorem (Theorem 1.5.1) and Corollaries 5.6.2 and 5.6.6. However, as pointed out by Bermond and Thomassen in [115], Theorem 6.5.2 does not imply Meyniel's theorem (Theorem 5.6.7). The following result is due to Häggkvist:

Theorem 6.5.3 *[391] Every hamiltonian digraph on n vertices and at least $\frac{1}{2}n(n+1) - 1$ arcs is pancyclic.* □

Song [679] generalized the result of Jackson given in Theorem 5.12.5 and proved the following theorem.

Theorem 6.5.4 *[679] Let $D = (V, A)$ be an oriented graph on $n \geq 9$ vertices with minimum degree $n-2$. Suppose that D satisfies the following property:*

$$xy \notin A \Rightarrow d^+(x) + d^-(y) \geq n - 3. \tag{6.2}$$

Then D is pancyclic. □

Song [679] pointed out that, if the minimum degree condition in Theorem 6.5.4 is relaxed, then it is no longer guaranteed that D is hamiltonian.

Using Theorem 6.5.4 and Theorem 10.7.3, Bang-Jensen and Guo proved that under the same conditions as in Theorem 6.5.4 the digraph is in fact vertex-pancyclic.

Theorem 6.5.5 *[54] Let D be an oriented graph on $n \geq 9$ vertices and suppose that D satisfies the conditions in Theorem 6.5.4. Then D is vertex pancyclic.* □

It should be noted that every digraph which satisfies the condition of Theorem 6.5.4 is a multipartite tournament with independence number at most 2.

There are several other results on pancyclicity of digraphs with large minimum degrees, see e.g. the papers [174, 175, 178] by Darbinyan.

6.5.2 Pancyclicity in Extended Semicomplete and Quasi-Transitive Digraphs

In this subsection we show how to use the close relationship between the class of quasi-transitive digraphs and the class of extended semicomplete digraphs to derive results on pancyclic and vertex-pancyclic quasi-transitive digraphs from analogous results for extended semicomplete digraphs.

A digraph D is **triangular with partition** V_0, V_1, V_2, if the vertex set of D can be partitioned into three disjoint sets V_0, V_1, V_2 with $V_0 \mapsto V_1 \mapsto V_2 \mapsto V_0$. Note that this is equivalent to saying that $D = \vec{C}_3[D\langle V_0\rangle, D\langle V_1\rangle, D\langle V_2\rangle]$.

Gutin [367] characterized pancyclic and vertex-pancyclic extended semicomplete digraphs. Clearly no extended semicomplete digraph of the form $D = \vec{C}_2[\overline{K}_{n_1}, \overline{K}_{n_2}]$ with at least 3 vertices is pancyclic since all cycles are of even length. Hence we must assume that there are at least 3 partite sets in order to get a pancyclic extended semicomplete digraph. It is also easy to see that the (unique) strong 3-partite extended semicomplete digraph on 4 vertices is not pancyclic (since it has no 4-cycle). These observations and the following theorem completely characterize pancyclic and vertex-pancyclic extended semicomplete digraphs.

Theorem 6.5.6 *[367] Let D be a hamiltonian extended semicomplete digraph of order $n \geq 5$ with k partite sets ($k \geq 3$). Then*

1. *(a) D is pancyclic if and only if D is not triangular with a partition V_0, V_1, V_2, two of which induce digraphs with no arcs, such that either $|V_0| = |V_1| = |V_2|$ or no $D\langle V_i\rangle$ ($i = 0, 1, 2$) contains a path of length 2.*
2. *(b) D is vertex-pancyclic if and only if it is pancyclic and either $k > 3$ or $k = 3$ and D contains two cycles Z, Z' of length 2 such that $Z \cup Z'$ has vertices in the three partite sets.* □

It is not difficult to see that Theorem 6.5.6 extends Theorem 1.5.1, since no semicomplete digraph on $n \geq 5$ vertices satisfies any of the exceptions from (a) and (b).

The next two lemmas by Bang-Jensen and Huang [79] concern cycles in triangular digraphs. They are used in the proof of Theorem 6.5.9 which characterizes pancyclic and vertex-pancyclic quasi-transitive digraphs.

Lemma 6.5.7 *[79] Suppose that D is a triangular digraph with a partition V_0, V_1, V_2 and suppose that D is hamiltonian. If $D\langle V_1\rangle$ contains an arc xy and $D\langle V_2\rangle$ contains an arc uv, then every vertex of $V_0 \cup \{x, y, u, v\}$ is on cycles of lengths $3, 4, \ldots, n$.*

Proof: Let C be a hamiltonian cycle of D. We construct an extended semicomplete digraph D' from D in the following way. For each of $i = 0, 1, 2$, first path-contract[5] each maximal subpath of C which is contained in $D\langle V_i\rangle$ and then delete the remaining arcs of $D\langle V_i\rangle$. It is clear that D' is a subdigraph of D, and in this process, C is changed to a hamiltonian cycle C' of D'. Hence D' is also triangular with a partition V'_0, V'_1, V'_2 such that $|V'_0| = |V'_1| = |V'_2| = r$, for some r (the last fact follows from the existence of a hamiltonian cycle in D'). Then each vertex of D is on a cycle of length k with $3r \leq k \leq |V(D)|$ (to see this, just use suitable pieces of the r subpaths of C in each V_i).

[5] Recall the definition of path-contraction from Subsection 5.1.1.

Now we may assume that $r \geq 2$ and we show that each vertex of $V_0 \cup \{x, y, u, v\}$ is on a cycle of length k with $3 \leq k \leq 3r-1$. To see this, we modify D' to another digraph D'' as follows. If x and y are in distinct maximal subpaths P_x, P_y of C in $D\langle V_1\rangle$, then we add (in D') an arc from the vertex to which P_x was contracted to the vertex to which P_y was contracted. If x and y are in the same maximal subpath P of C in $D\langle V_1\rangle$, then we add (in D') an arc from the vertex to which P was contracted to an arbitrary other vertex of V_1'. For the vertices u and v we make a similar modification. Hence we obtain a digraph D'' which is isomorphic to a subdigraph of D. The digraph D'' is also triangular with a partition V_0'', V_1'', V_2'' such that $|V_0''| = |V_1''| = |V_2''| = r$. Moreover $D''\langle V_1''\rangle$ contains an arc $x'y'$ and $D''\langle V_2''\rangle$ contains an arc $u'v'$. It is clear now that each vertex of $V_0'' \cup \{x', y', u', v'\}$ is on a cycle of length k where $3 \leq k \leq 3r-1$. Using the same structure as for these cycles we can see that in D each vertex of $V_0 \cup \{x, y, u, v\}$ is on a cycle of length k with $3 \leq k \leq 3r-1$. □

Lemma 6.5.8 *[79] Suppose that D is a triangular digraph with a partition V_0, V_1, V_2 and D has a hamiltonian cycle C. If $D\langle V_0\rangle$ contains an arc of C and a path P of length 2, then every vertex of $V_1 \cup V_2 \cup V(P)$ is on cycles of lengths $3, 4, \ldots, n$.*

Proof: Exercise 6.24. □

It is easy to check that a strong quasi-transitive digraph on 4 vertices is pancyclic if and only if it is a semicomplete digraph. For $n \geq 5$ we have the following characterization due to Bang-Jensen and Huang:

Theorem 6.5.9 *[79] Let $D = (V, A)$ be a hamiltonian quasi-transitive digraph on $n \geq 5$ vertices.*

1. *(a) D is pancyclic if and only if it is not triangular with a partition V_0, V_1, V_2, two of which induce digraphs with no arcs, such that either $|V_0| = |V_1| = |V_2|$, or no $D\langle V_i\rangle$ ($i = 0, 1, 2$) contains a path of length 2.*
2. *(b) D is not vertex-pancyclic if and only if D is not pancyclic or D is triangular with a partition V_0, V_1, V_2 such that one of the following occurs:*
   - *(b1) $|V_1| = |V_2|$, both $D\langle V_1\rangle$ and $D\langle V_2\rangle$ have no arcs, and there exists a vertex $x \in V_0$ such that x is not contained in any path of length 2 in $D\langle V_0\rangle$ (in which case x is not contained in a cycle of length 5).*
   - *(b2) one of $D\langle V_1\rangle$ and $D\langle V_2\rangle$ has no arcs and the other contains no path of length 2, and there exists a vertex $x \in V_0$ such that x is not contained in any path of length 1 in $D\langle V_0\rangle$ (in which case x is not contained in a cycle of length 5).*

Proof: To see the necessity of the condition in (a), suppose that D is triangular with a partition V_0, V_1, V_2, two of which induce digraphs with no arcs. If $|V_0| = |V_1| = |V_2|$, then D contains no cycle of length $n-1$. If no $D\langle V_i\rangle$