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Modified Chebyshev-2 Filters with Low Q-Factors

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ABSTRACT. A modified low pass maximally flat inverse Chebyshev filter is suggested in this paper, and is shown to have improvement over previous known Chebyshev filters. The coefficients of this modified filter, using a higher order polynomial with multiplicity of the dominant pole-pair, have been determined. The pole locations and the Q factors for different orders of filters are tabulated using an optimization algorithm.

1. Introduction

Recently, there has been a great deal of interest in deriving suboptimal transfer functions with low Q dominant poles[1-3]. The suboptimality of these functions allows for low precision requirements in both active RC-filters and digital filters. It is, however, noted that the minimal order filter is not necessarily the least complex filter.

In a recent article, Premoli[1] used the notion of multiplicity in the dominant poles to obtain multiple critical root maximally flat (MUCROMAF) polynomials for low-pass filters. An alternative method for deriving these functions, referred to as modified Butterworth functions, has been presented by Massad and Yarlagadda[2]. Also a new class of multiple critical root pair, equal ripple (MUCROER) filtering functions, having higher degree than the filtering functions, has been presented by Premoli[1]. By relaxing the equal ripple conditions, Massad and Yarlagadda derived a new algorithm to find modified Chebyshev functions which have lower Q dominant poles[2]. The generation of modified functions involves starting with a classical function such as Butterworth of Chebyshev of order n and deriving a suboptimal function of order $m = n + 2(c-1)$, where c corresponds to the multiplicity of the dominant pole pair. It is known that the rate drop of Q_d (the quality factor of the dominant multiple poles)[2] is largest when $c = 2$, which corresponds to $m = n + 2$. The Q_d is given by

$$Q_d = \frac{\sqrt{R^2 + I^2}}{2R} \tag{1}$$

where R and I are the real and imaginary parts of the dominant poles.

In this paper, the same multiple dominant pole notion is used to develop modified Chebyshev-2 filtering functions. The derived function has the maximally flat property in the pass-band and the non equiripple property in the stop-band and the proposed approach is an extension of an earlier paper on modified Butterworth function[2]. Poles for the modified functions and previous Chebyshev-2 functions are given in Tables 1 and 2 respectively, while the Q's of the dominant poles are com-

TABLE 1. Poles of modified Chybyshev-2 transfer function $L_m(S)$.

For $c = 2$, $m = n + 2(c-1)$, Pass-Band Spec. 3dB					
m	Double Pole Column				
4	− .49699446 ± j .57476077				
5	− .33268994 ± j .60048171	− .57725896			
6	− .25147142 ± j .60611979	− .52886723 ± j .20605848			
7	− .20230990 ± j .60582930	− .46516686 ± j .32236163	− .56394148		
8	− .16919994 ± j .60327956	− .40688466 ± j .39083448	− .54578182 ± j .14471478		
9	− .14539173 ± j .59987729	− .35806911 ± j .43315928	− .50818486 ± j .24482882	− .56723780	
10	− .12742565 ± j .59615488	− .31810163 ± j .46044171	− .46632798 ± j .31408764	− .55683296 ± j .11300050	
11	− .11332476 ± j .59241414	− .28535770 ± j .46868511	− .42637940 ± j .36259098	− .53123961 ± j .19892716	− .57080646
12	− .10205094 ± j .58879143	− .25835009 ± j .49127497	− .39041116 ± j .39719556	− .49984378 ± j .26365734	− .56371039 ± j .092987759

pared with those of the Chebyshev and modified Chebyshev functions in Table 3. Table 4 shows the Chebyshev polynomial of degree $2n$ of order 2 to 10, while Table 5 shows the new generated Inverse Chebyshev polynomial of Table 4.

2. Problem Statement

Let

$$|H_n(j\omega)|^2 = \frac{C_n^2\left(\frac{\omega_r}{\omega}\right)}{C_n^2\left(\frac{\omega_r}{\omega}\right) + \varepsilon^2 C_n^2\left(\frac{\omega_r}{\omega_c}\right)} \tag{2}$$

be the Chebyshev-2 function (see Appendix A for definition) satisfying the pass-

TABLE 2. Poles of Chybyshev-2 transfer function $H_n(S)$.

For $c = 2$, $m = n + 2(c-1)$, Pass-Band Spec. 3dB					
n	Single Pole Column				
2	− .40000000 ± j .48989795				
3	− .24796591 ± j .51365275	− .65598996			
4	− .18106360 ± j .51828533	− .55552149 ± j .27282695			
5	− .14307318 ± j .52004107	− .44731512 ± j .38382219	− .62832847		
6	− .11841435 ± j .52077428	− .36910064 ± j .43495310	− .58690160 ± j .18531705		
7	− .10105240 ± j .52095341	− .31304690 ± j .46189562	− .52097939 ± j .29521252	− .62010201	
8	− .088141864 ± j .520280228	− .27150485 ± j .47757044	− .45939046 ± j .37076716	− .59700732 ± j .13957058	
9	− .078156206 ± j .520453044	− .23963385 ± j .48735523	− .40763603 ± j .40162704	− .55375153 ± j .23665607	− .61539617
10	− .070199156 ± j .51998829	− .21444991 ± j .49378078	− .36511099 ± j .42835031	− .50729719 ± j .30325120	− .60043964 ± j .11157222

TABLE 3. Comparison of quality factors.

For $c = 2$, $m = n + 2(c-1)$, Pass-Band Spec. 3dB			
n	Chebyshev-2 Q_i	m	MCF-2 Q_i
2	0.7905694165	4	0.7644328365
3	1.150105215	5	1.031718055
4	1.516048694	6	1.304752092
5	1.884920671	7	1.578559034
6	2.255078176	8	1.851531200
7	2.625686159	9	2.122696745
8	2.996353102	10	2.392066040
9	3.366902594	11	2.661183131
10	3.73726279	12	2.927801861

TABLE 4. Chebyshev polynomial of degree $2n$ of order 2 to 10.

n	$C_n^2(\omega)$
2	$1-4\omega^2+4\omega^4$
3	$9\omega^2-24\omega^4+16\omega^6$
4	$1-16\omega^2+80\omega^4-128\omega^6+64\omega^8$
5	$25\omega^2-200\omega^4+560\omega^6-640\omega^8+256\omega^{10}$
6	$1-36\omega^2+420\omega^4-1792\omega^6+3456\omega^8-3072\omega^{10}+1024\omega^{12}$
7	$49\omega^2-784\omega^4+4704\omega^6-13440\omega^8+19712\omega^{10}-14336\omega^{12}+4096\omega^{14}$
8	$1-64\omega^2+1344\omega^4-10752\omega^6+42240\omega^8-90112\omega^{10}+106496\omega^{12}-65536\omega^{14}+16384\omega^{16}$
9	$81\omega^2-2160\omega^4+22176\omega^6-114048\omega^8+329472\omega^{10}-559104\omega^{12}+552960\omega^{14}-294912\omega^{16}+65536\omega^{18}$
10	$1-100\omega^2+3300\omega^4-42240\omega^6+274560\omega^8-1025024\omega^{10}+2329600\omega^{12}-3276800\omega^{14}+2785280\omega^{16}$ $-1310720\omega^{18}+262144\omega^{20}$

TABLE 5. Inverse Chebyshev polynomial of degree $2n$ of order 2 to 10.

n	$C_n^2\left(\frac{1}{\omega}\right)$
2	$(4-4\omega^2+\omega^4)\,\omega^{-4}$
3	$(16-24\omega^2+9\omega^4)\,\omega^{-6}$
4	$(64-128\omega^2+80\omega^4-16\omega^6+\omega^8)\,\omega^{-8}$
5	$(256-640\omega^2+560\omega^4-200\omega^6+25\omega^8)\,\omega^{-10}$
6	$(1024-3072\omega^2-3456\omega^4-1792\omega^6+420\omega^8-36\omega^{10}+\omega^{12})\,\omega^{-12}$
7	$(4096-14336\omega^2+19712\omega^4-13440\omega^6+4704\omega^8-784\omega^{10}+49\omega^{12})\,\omega^{-14}$
8	$(16384-65536\omega^2+106496\omega^4-90112\omega^6+42240\omega^8-10752\omega^{10}+1344\omega^{12}-64\omega^{14}+\omega^{16})\,\omega^{-16}$
9	$(65536-294912\omega^2+552960\omega^4-559104\omega^6+329472\omega^8-114048\omega^{10}-22176\omega^{12}-2160\omega^{14}$ $+81\omega^{16})\,\omega^{-18}$
10	$(262144-1310720\omega^2+2785280\omega^4-3276800\omega^6+2329600\omega^8-1025024\omega^{10}+274560\omega^{12}$ $-42240\omega^{14}+3300\omega^{16}-100\omega^{18}+\omega^{20})\,\omega^{-20}$

band and stop-band requirements in the frequency domain, where it is assumed that $0<|\omega|<\omega_c$ corresponds to the pass-band, and $\omega_c<|\omega|<\omega_r$ corresponds to the transition region, while $|\omega|>\omega_r$ corresponds to the stop-bands, and $C_n\left(\frac{\omega_r}{\omega_c}\right)$ is a number which takes care of the stop-band specification, as shown by Fig. 1.

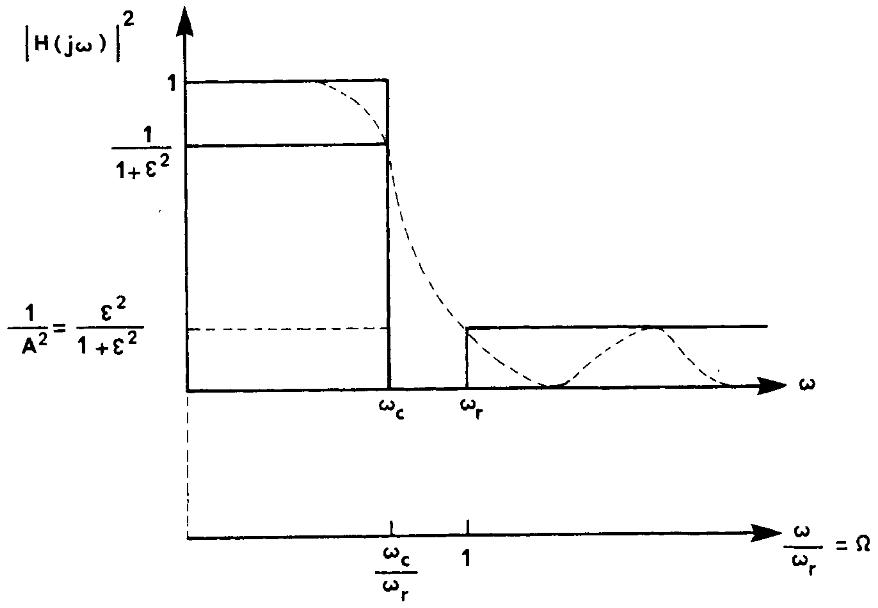

FIG. 1. Low-pass filter specification.

Defining a normalized frequency $\Omega \equiv \frac{\omega}{\omega_r}$, then it is clear from Fig. 1 that $\frac{\omega_c}{\omega_r} = \Omega_c$ and the beginning of the stop-band Ω_r is unity. Consequently, (2) can be expressed as

$$|H_n(j\Omega)|^2 = \frac{C_n^2(\frac{1}{\Omega})}{C_n^2(\frac{1}{\Omega}) + \varepsilon^2 C_n^2(\frac{1}{\Omega_c})} \tag{3}$$

It is required to find a modified Chebyshev-2 function of the form

$$|L_m(j\Omega)|^2 = \frac{C_n^2(\frac{1}{\Omega})}{C_n^2(\frac{1}{\Omega}) + \varepsilon^2 C_n^2(\frac{1}{\Omega_c}) \sum_{i=1}^{m} d_{2i} \frac{\Omega^{2i}}{\Omega^{2n}}} \tag{4}$$

satisfying the frequency domain specifications with the constraint that

$$|F(j\Omega)|^2 = \left|\frac{H_n(j\Omega)}{L_m(j\Omega)}\right|^2 \tag{5}$$

deviates the least amount from unity for frequencies close to $\Omega = 0$. Also, the Q factor of the dominant poles from (4) must be less than the Q factor of the dominant poles of (1). As the function of (3) is replaced by the function of (4), the new function can be easily shown to have the same zeros and poles as (3) in addition to three additional poles.

3. Modified Chebyshev-2 Polynomials

The maximum flatness in the pass-band imposes the constraint that $|F(j\Omega)|^2 = |H_n(j\Omega) / L_m(j\Omega)|^2$ should deviates the least amount from unity at frequencies close to zero. It is required that the first $(n-1)$ derivatives of (4) with respect to Ω^2 must be equal zero at $\Omega = 0$[2]. This results in $d_2 = d_4 = \ldots = d_{2n-2}$.

Noting that

$$\Omega^{-2n} \sum_{i=1}^{m} d_{2i}\, \Omega^{2i} = \sum_{i=1}^{n+2} d_{2i} \frac{\Omega^{2i}}{\Omega^{2i}}$$

where $m = n + 2(c-1)$ and $c = 2$ for multiplicity of the pole.

Therefore equation (4) can be rewritten as

$$|L_m(j\Omega)|^2 = \frac{C_n^2\left(\frac{1}{\Omega}\right)}{C_n^2\left(\frac{1}{\Omega}\right) + \varepsilon^2 C_n^2\left(\frac{1}{\Omega_c}\right) \sum_{i=n}^{n+2} d_{2i} \frac{\Omega^{2i}}{\Omega^{2n}}} \tag{6}$$

$$= \frac{1}{1 + \varepsilon^2 \dfrac{C_n^2\left(\frac{1}{\Omega_c}\right)}{C_n^2\left(\frac{1}{\Omega}\right)} \left[d_{2n} + d_{2(n+1)}\Omega^2 + d_{2(n+2)}\Omega^4\right]} \tag{7}$$

Or in general form

$$|L_m(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \dfrac{C_n^2\left(\frac{1}{\Omega_c}\right)}{C_n^2\left(\frac{1}{\Omega}\right)} \left[\sum_{i=0}^{2(c-1)} d_{2(1+n)}\Omega^{2i}\right]} \tag{8}$$

At the cutoff frequency, $\Omega = 1$, (normalized), it is required that $L_m|(j\Omega)|^2 \le |H_n(j\Omega)|^2$, consequently

$$\sum_{i=0}^{2(c-1)} d_{2(i+n)}\,\Omega^{2i} = \sum_{i=0}^{2(c-1)} d_{2(i+n)} = 1, \text{ at } \Omega = 1$$

therefore $\quad d_{2(n)} + d_{2(1+n)} + \ldots + d_{2(2c-2+n)} = 1$

or

$$d_{2(n)} = 1 - \sum_{i=1}^{2(c-1)} d_{2(i+n)} \tag{9}$$

where the d_i's are the unknown coefficients to be determined.

It follows from (3) and (4), that

$$|L_m(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 \dfrac{C_n^2(\frac{1}{\Omega_c})}{C_n^2(\frac{1}{\Omega})}\left[1-\sum_{i=1}^{2(c-1)} d_{2(i+n)} + \sum_{i=1}^{2(c-1)} d_{2(i+n)}\Omega^{2i}\right]}$$

$$= \frac{1}{G(\Omega)} \tag{10}$$

Assuming the pole multiplicity, it follows that the first $(c-1)$ derivatives of $G(\Omega)$ will be zero at $\Omega = \Omega_d = (R+jI)$; in addition, it is clear that $G(\Omega_d) = 0$. These give a set of $2c$ simultaneous nonlinear equations in $2c$ unknowns R, I, $d_{2(2c+n-2)}$.

A look into the pass- and stop-band specifications reveals that the function $|L_m(j\Omega)|^2$ satisfies the stop-band specification. That is,

$$|L_m(j\Omega)|^2 \le |H_n(j\Omega)|^2 \quad \text{for } \Omega \ge 1$$

Using an argument similar to the presentation of Premoli[1], one can show that the above equation is true. However, the pass-band specifications may not be met, in general. Note that the constraints are reversed from the modified Butterworth case. If pass-band specifications need to be satisfied, relevant modifications are discussed in the next section.

Looking briefly into the solution of the above mentioned equations, we will consider $c = 2$, as it corresponds to the largest Q_d drop. The four equations involving four unknowns can be reduced to two equations in two unknowns by eliminating $d_{2(2c+n-3)}$ and $d_{2(c+n-2)}$. The real and imaginary parts of the equation give two equations in two unknowns, the real and imaginary parts of Ω_d.

4. Computer Method to Solve Nonlinear Simultaneous Equation

To find the pole-locations of modified filter function two computer programs are used, one of which solves a system of simultaneous nonlinear equations[4]. These simultaneous equations are a result of some constraints on the dominant poles. The computer program calls several subroutines to find a local minimum of a function which can be expressed as a sum of squares of functions. The following methods are used in solving the set of equations

1. Marquardt's Method ("downhill" method);
2. The Guass-Newton Method ("one-step" method).

The above nonlinear least squares problem is defined as

$$\text{Minimize } PHI = \sum_{i=1}^{N} \frac{(\text{fitted model})_i^2 - (\text{observed values})_i^2}{(\text{standard error})_i^2}$$

where PHI is the function to be minimized.

The computer program is available on request from any of the authors.

5. Performance of The Modified Filter

Variation of the magnitude squared for the modified and the original inverse Chebyshev filter, versus the frequency are shown in Fig. 2 and 3, respectively. It is

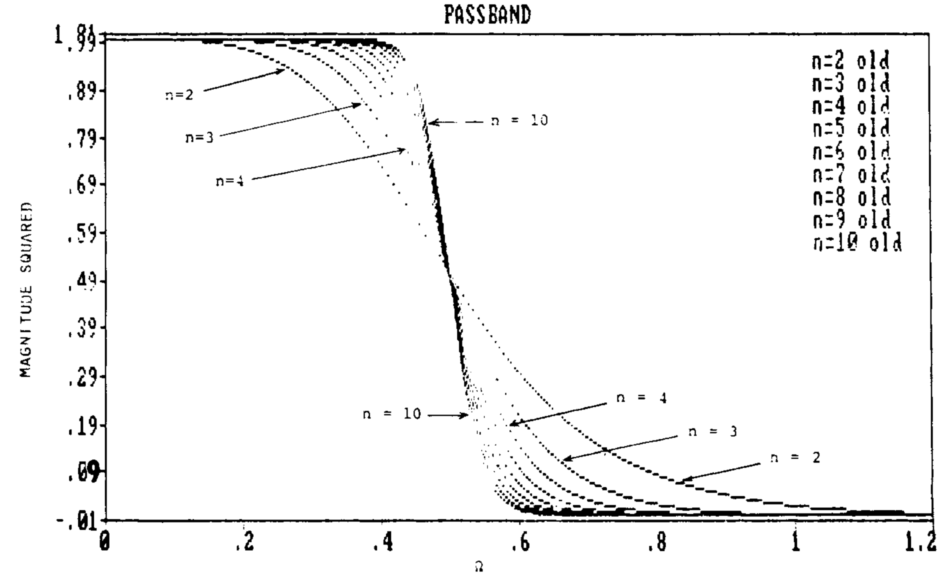

FIG. 2. Variation of the magnitude squared of the original inverse Chebyshev function against frequency for differing values of n (different order).

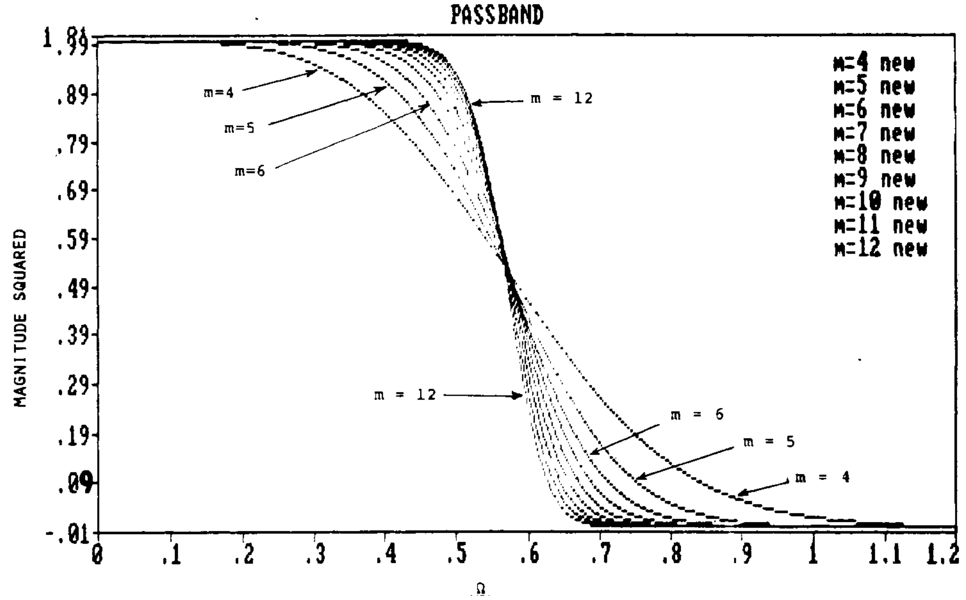

FIG. 3. Variation of the magnitude squared of the modified inverse Chebyshev function against frequency for differing values of M (different order).

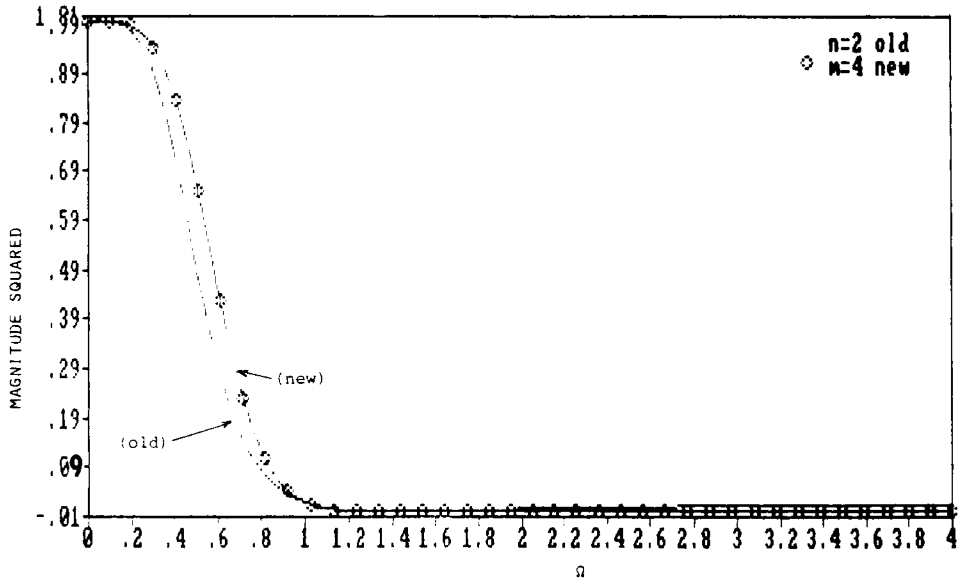

FIG. 4. Magnitude comparison between second order inverse Chebyshev and fourth order modified inverse Chebyshev function.

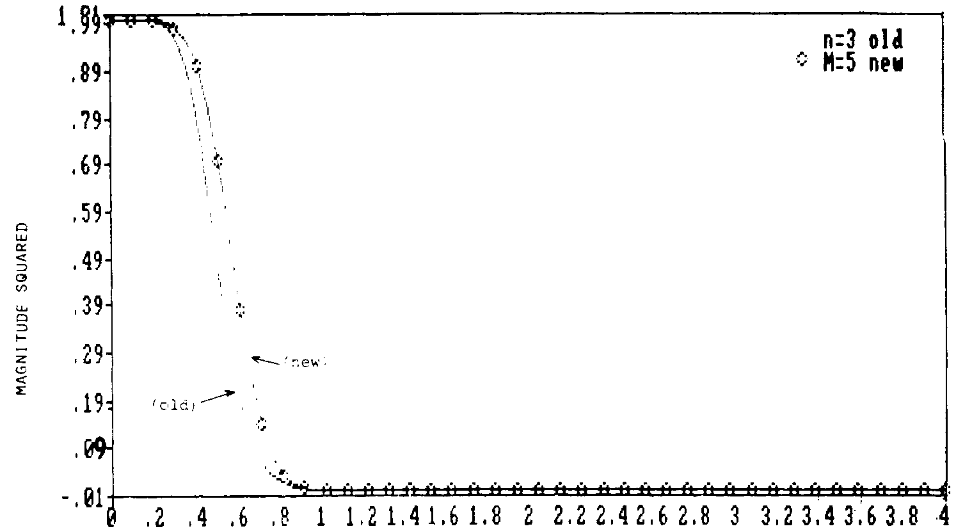

FIG. 5. Magnitude comparison between third order inverse Chebyshev and fifth order modified inverse Chebyshev function.

clear from the graphs that the modified filter provides both a wider bandwidth and a shorter transition region than the normal filter.

Magnitude comparisons between different order inverse Chebyshev and different order modified inverse Chebyshev function are shown in Fig. 2,3,4 and 5, respectively. Deduction of the differences of bandwidth and the transition region for both filters are obvious.

Figure 6 shows the performance of both filters in the band-stop region. It is clear from the graphs that the modified filter has a minimized side lobe which shows the ability of this filter to work as a perfect stop-band in the above mentioned region.

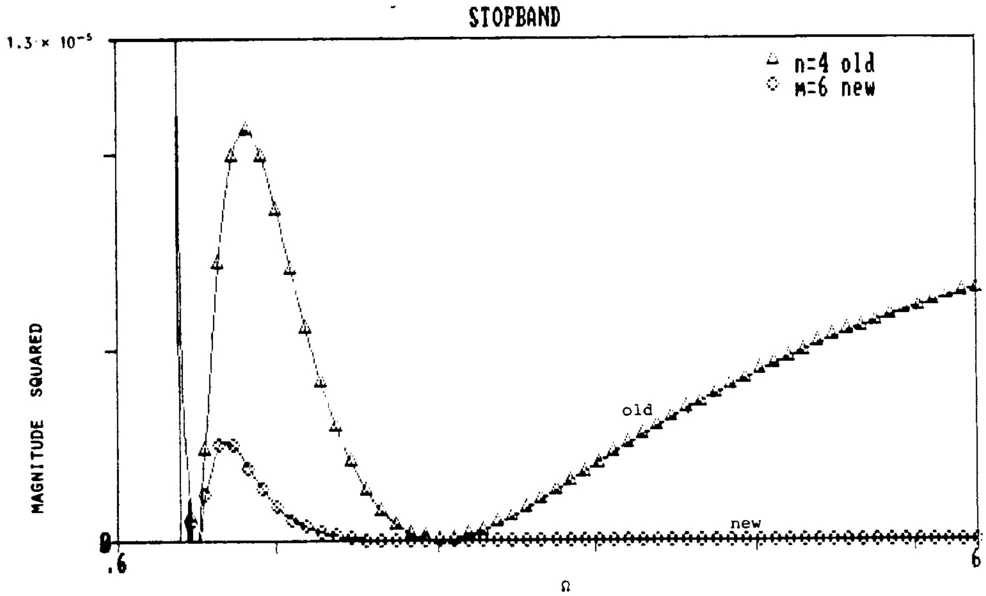

FIG. 6. Magnitude comparison between fourth order inverse Chebyshev and sixth order modified inverse Chebyshev function.

6. Conclusion

A modified maximally flat inverse Chebyshev function with a higher degree but much reduced dominant pole pair Q factor than that of the corresponding inverse Chebyshev function has been presented. The transfer function is derived for different orders using the assumption that the derivative of its magnitude squared with respect to the squared frequency variable is equal to zero. The modified transfer function gives a higher quality filter and therefore better stability on the expense of a reduced transition region. However, as the order of the filter increases, little deviation program using the Marquardt and modified Marquardt methods and the Gauss-Newton method has been used to determine the pole locations.

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Appendix A

Chebyshev-2 Functions

Inverse Chebyshev Filter

A Chebyshev polynomial of degree n is defined by

$$\begin{aligned} C_n(\omega) &= \cos(n \arccos \omega), \quad \omega \leq 1 \\ &= \cosh(n \operatorname{arc\,cosh} \omega), \quad \omega > 1 \end{aligned} \tag{A1}$$

The squared transfer function of an nth order type I characteristic filter is given by[5]

$$|H_I(j\omega)|^2_{LP} = \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \tag{A2}$$

as shown in Fig. 7a.

In the denominator of (A2), the square of the Chebyshev polynomial is multiplied by a ripple factor ε^2, since the polynomials take negative as well as positive values with a maximum absolute value of unity in the pass-band $-1 \leq \omega \leq 1$. The value of ε controls the pass- and stop-band specifications.

The resulting transfer function $H_I(j\omega)$ is characterised by equal ripple in the pass-band and by monotonic fall-off in the stop-band.

Subtracting such a response from unity, as illustrated in Fig. 7b gives a new squared magnitude function which is known as high-pass type I filter.

$$\begin{aligned} |H_I(j\omega)|^2_{HP} &= 1 - |H_I(j\omega)|^2_{HP} \\ &= 1 - \frac{1}{1 + \varepsilon^2 C_n^2(\omega)} \\ &= \frac{\varepsilon^2 C_n^2(\omega)}{1 + \varepsilon^2 C_n^2(\omega)} \end{aligned} \tag{A3}$$

Applying the frequency transformation, replacing ω by $\frac{1}{\omega}$ thus interchanging the

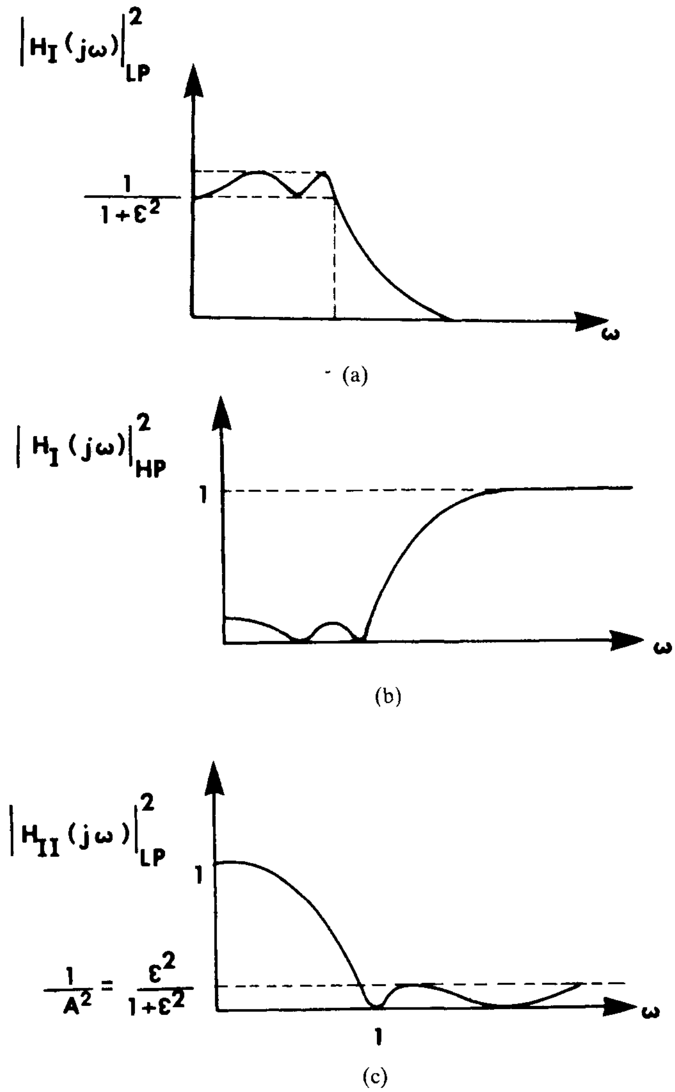

FIG. 7. The Chebyshev filter (a), and an intermediate step (b) to obtain the inverse Chebyshev response in (c).

behavior at infinity and the origion, the high-pass type I becomes a low-pass type II filter as shown by Fig. 7c give by

$$|H_{II}(j\omega)|^2_{LP} = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2(\frac{1}{\omega})}} \tag{A4}$$

For convenience in comparing the formulas for the value of n required for a specified attenuation in the pass- and stop-bands of a Chebyshev and inverse Chebyshev (*i.e.*, type I and type II) characteristic filter respectively, we redefine

$$\frac{1}{\varepsilon^2} = \varepsilon'^2 C_n^2 \left(\frac{1}{\omega_c} \right) \tag{A5}$$

Therefore (A5) becomes

$$| H_{II}(j\omega) |_{HP}^2 = \frac{1}{1 + \dfrac{\varepsilon'^2 C_n^2 \left(\frac{1}{\omega_c}\right)}{C_n^2 \left(\frac{1}{\omega}\right)}}$$

$$= \frac{C_n^2 \left(\frac{1}{\omega}\right)}{C_n^2 \left(\frac{1}{\omega}\right) + \varepsilon'^2 C_n^2 \left(\frac{1}{\omega_c}\right)} \tag{A6}$$

rewriting ε' as ε, we get

$$| H_{II}(j\omega) |_{LP}^2 = \frac{C_n^2 \left(\frac{1}{\omega}\right)}{C_n^2 \left(\frac{1}{\omega}\right) + \varepsilon^2 C_n^2 \left(\frac{1}{\omega_c}\right)} \tag{A7}$$

مرشحات شبيشيف الثنائية المطورة ذات عوامل النوعية المنخفضة

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جـــدة - المملكة العربية السعودية .

المستخلص . تعرض الورقة مرشحات تردد شبيشيف المعكوسة (الثنائية) ذات الخواص المسطحة ، والتي تبين تَمَيُّزها على سابقاتها من مرشحات شبيشيف . حددت الورقة معاملات المرشحات المطورة وذلك باستخدام تعابير رياضية تحوى متغيرات ذات أسس مرتفعة ، مع تكرار الأقطاب المزدوجة المتقاربة من المحور غير الحقيقي ، كما صَنفت الورقة مواقع هذه الأقطاب وعوامل النوعية لمختلف المرشحات ، وذلك باستخدام طريقة مثلي مرتبة .