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Solution of Linear and Nonlinear Partial Differential Equations Using Mixture of Elzaki Transform and the Projected Differential Transform Method

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Abstract

The aim of this study is to solve some linear and nonlinear partial differential equations using the new integral transform "Elzaki transform" and projected differential transform method. The nonlinear terms can be handled by using of projected differential transform method; this method is more efficient and easy to handle such partial differential equations in comparison to other methods. The results show the efficiency and validation of this method.
Keywords: Elzaki transform, projected differential transform method, nonlinear partial differential equations.

1. Introduction

Many problems of physical interest are described by linear and nonlinear partial differential equations with initial or boundary conditions, these problems are fundamental importance in science and technology especially in engineering. Some valuable contributions have already been made to showing differential equations such as Laplace transform method [lslam, Yasir Khan, Naeem Faraz and Francis Austin (2010), Lokenath Debnath and Bhatta (2006)], Sumudu transform [A.Kilicman and H.E.Gadain (2009), (2010)], differential transform method etc. Elzaki transform method is very effective tool for solve linear partial differential equations [Tarig Elzaki & Salih Elzaki (2011)]. In this study we use the projected differential transform method [Nuran Guzel and Muhammet Nurulay (2008), Shin- Hsiang Chang , Ling Chang (2008)] to decompose the nonlinear terms, this means that we can use both Elzaki transform and projected differential transform methods to solve many nonlinear partial differential equations.

1.1. Elzaki Transform:

The basic definitions of modified of Sumudu transform or Elzaki transform is defined as follows, Elzaki transform of the function $f(t)$ is

$$E[f(t)] = v\int_0^\infty f(t)\, e^{-\frac{t}{v}} dt\ , \qquad t > 0$$

Tarig M. Elzaki and Sailh M. Elzaki in [2011], showed the modified of Sumudu transform [Kilicman & ELtayeb (2010)] or Elzaki transform was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations.

In this paper, we combined Elzaki transform and projected differential transform methods to solve linear and nonlinear partial differential equations.

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - vf(x,0) \qquad E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}$$

$$E\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}[T(x,v)] \qquad E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$$

Proof:

To obtain ELzaki transform of partial derivatives we use integration by parts as follows:

$$\mathrm{E}\left[\frac{\partial f}{\partial t}(x,t)\right] = \int_0^\infty v\frac{\partial f}{\partial t}e^{\frac{-t}{v}}dt = \lim_{p\to\infty}\int_0^p ve^{\frac{-t}{v}}\frac{\partial f}{\partial t}dt = \lim_{p\to\infty}\left\{\left[ve^{\frac{-t}{v}}f(x,t)\right]_0^p - \int_0^p e^{\frac{-t}{v}}f(x,t)dt\right\}$$

$$= \frac{T(x,v)}{v} - vf(x,0)$$

We assume that f is piecewise continuous and it is of exponential order.

Now $\mathrm{E}\left[\frac{\partial f}{\partial x}\right] = \int_0^\infty ve^{\frac{-t}{v}}\frac{\partial f(x,t)}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty ve^{\frac{-t}{v}}f(x,t)dt,$ using the Leibnitz rule to find:

$$\mathrm{E}\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}\left[T(x,v)\right]$$

By the same method we find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[T(x,v)\right]$

To find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$

Let $\frac{\partial f}{\partial t} = g\ ,\ then$ we have:

$$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] = \mathrm{E}\left[\frac{\partial g(x,t)}{\partial t}\right] = \mathrm{E}\frac{\left[g(x,t)\right]}{v} - vg(x,0)$$

$$E\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f}{\partial t}(x,0)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

1.2. Projected Differential Transform Methods:

In this section we introduce the projected differential transform method which is modified of the differential transform method.

Definition:

The basic definition of projected differential transform method of function $f(x_1,x_2,\cdots\cdots,x_n)$ is defined as

$$f(x_1,x_2,\cdots\cdots,x_{n-1},k)=\frac{1}{k!}\left[\frac{\partial^k f(x_1,x_2,\cdots\cdots,x_n)}{\partial x_n^k}\right]_{x_n=0} \quad (1)$$

Such that $f(x_1,x_2,\cdots\cdots,x_n)$ is the original function and $f(x_1,x_2,\cdots\cdots,x_{n-1},k)$ is projected transform function.

And the differential inverse transform of $f(x_1,x_2,\cdots\cdots,x_{n-1},k)$ is defined as

$$f(x_1,x_2,\cdots\cdots,x_n)=\sum_{k=o}^{\infty}f(x_1,x_2,\cdots\cdots,x_n)(x-x_0)^k \quad (2)$$

The fundamental theorems of the projected differential transform are

Theorems:

$$(1)\quad If\ z(x_1,x_2,......,x_n)=u(x_1,x_2,......,x_n)\pm v(x_1,x_2,......,x_n)$$
$$Then\ \ z(x_1,x_2,......,x_{n-1},k)=u(x_1,x_2,......,x_{n-1},k)\pm v(x_1,x_2,......,x_{n-1},k)$$

$$(2)\quad If z(x_1,x_2,......,x_n)=c\,u(x_1,x_2,......,x_n)$$
$$Then\ \ z(x_1,x_2,......,x_{n-1},k)=cu(x_1,x_2,......,x_{n-1},k)$$

$$(3)\quad If z(x_1,x_2,......,x_n)=\frac{du(x_1,x_2,......,x_n)}{dx_n}$$
$$Then\ z(x_1,x_2,......,x_{n-1},k)=(k+1)u(x_1,x_2,.....,x_{n-1},k+1)$$

$$(4)\ If z(x_1,x_2,......,x_n)=\frac{d^n u(x_1,x_2,......,x_n)}{dx_n^n}$$
$$Then\ z(x_1,x_2,......,x_{n-1},k)=\frac{(k+n)!}{k!}u(x_1,x_2,.....,x_{n-1},k+n)$$

$$(5)\ If\quad z(x_1,x_2,......,x_n)=u(x_1,x_2,......,x_n)\ v(x_1,x_2,......,x_n)$$
$$Then\quad z(x_1,x_2,......,x_{n-1},k)=\sum_{m=0}^{k}u(x_1,x_2,......,x_{n-1},m)\ v(x_1,x_2,......,x_{n-1},k-m)$$

$$(6)\ If\ \ z(x_1,x_2,......,x_n)=u_1(x_1,x_2,......,x_n)u_2(x_1,x_2,......,x_n).....u_n(x_1,x_2,......,x_n)\,Then$$
$$z(x_1,x_2,......,x_{n-1},k)=\sum_{k_{n-1}=0}^{k}\ \sum_{k_{n-2}=0}^{k_{n-1}}\ ......\ \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}u_1(x_1,x_2,......,x_{n-1},k_1)u_2(x_1,x_2,......,x_{n-1},k_2-k_1)$$
$$\times.....u_{n-1}(x_1,x_2,......,x_{n-1},k_{n-1}-k_{n-2})u_n(x_1,x_2,......,x_{n-1},k-k_{n-1})$$

$$(7)\quad If\quad z(x_1,x_2,......,x_n)=x_1^{q_1}x_2^{q_2}........x_n^{q_n}$$
$$Then\quad z(x_1,x_2,......,x_{n-1},k)=\delta(x_1,x_2,......,x_{n-1},q_n-k)=\begin{cases}1 & k=q_n\\ 0 & k\neq q_n\end{cases}$$

Note that c is a constant and n is a nonnegative integer.

2. Applications:

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{3}$$

$$u(x,0)=h(x)\quad,\quad u_t(x,0)=f(x)$$

Where D is linear differential operator of order two, R is linear differential operator of less order than D, N is the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking Elzaki transform on both sides of equation (3), to get:

$$E[Du(x,t)]+E[Ru(x,t)]+E[Nu(x,t)]=E[g(x,t)] \tag{4}$$

Using the differentiation property of Elzaki transforms and above initial conditions, we have:

$$E[u(x,t)]=v^2E[g(x,t)]+v^2h(x)+v^3f(x)-v^2E[Ru(x,t)+Nu(x,t)] \tag{5}$$

Appling the inverse Elzaki transform on both sides of equation (5), to find:

$$u(x,t)=G(x,t)-E^{-1}\left\{v^2E\left[Ru(x,t)+Nu(x,t)\right]\right\} \tag{6}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the projected differential transform method.

$$u(x,m+1)=-E^{-1}\left\{v^2E\left[A_m+B_m\right]\right\}\ ,\ u(x,0)=G(x,t) \tag{7}$$

Where A_m, B_m are the projected differential transform of $Ru(x,t)$, $Nu(x,t)$.

From equation (7), we have:

$$u(x,0)=G(x,t)\quad ,\quad u(x,1)=-E^{-1}\left\{v^2E\left[A_0+B_0\right]\right\}$$
$$u(x,2)=-E^{-1}\left\{v^2E\left[A_1+B_1\right]\right\}\quad ,\quad u(x,3)=-E^{-1}\left\{v^2E\left[A_2+B_2\right]\right\}$$
$$.....$$

Then the solution of equation (3) is

$$u(x,t)=u(x,0)+u(x,1)+u(x,2)+........$$

To illustrate the capability and simplicity of the method, some examples for different linear and nonlinear partial differential equations will be discussed.

Example 2.1:

Consider the simple first order partial differential equation

$$\frac{\partial y}{\partial x}=2\frac{\partial y}{\partial t}+y\quad ,\quad y(x,0)=6e^{-3x} \tag{8}$$

Taking Elzaki transform of (8), leads to

$$E\left[y(x,t)\right]=6v^2e^{-3x}+\frac{v}{2}E\left[A_m-B_m\right]$$

Take the inverse Elzaki transform to find,

$$y(x,m+1)=E^{-1}\left\{\ \frac{v}{2}E\left[A_m-B_m\right]\ \right\}\quad ,\quad y(x,0)=6e^{-3x} \tag{9}$$

Where $A_m = \dfrac{\partial y(x,m)}{\partial x}$, $B_m = y(x,m)$ are the projected differential transform of $\dfrac{\partial y(x,t)}{\partial x}$, $y(x,t)$.

The standard Elzaki transform defines the solution $y(x,t)$ by the series

$$y(x,t) = \sum_{m=0}^{\infty} y(x,m)$$

From equation (9) we find that: $y(x,0) = 6e^{-3x}$

$$A_0 = -18e^{-3x},\ B_0 = 6e^{-3x},\ y(x,1) = E^{-1}\left[-12v^3 e^{-3x}\right] = -12te^{-3x}$$

$$A_1 = 36te^{-3x},\ B_1 = -12e^{-3x},\ y(x,2) = E^{-1}\left[24v^4 e^{-3x}\right] = 12t^2 e^{-3x}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad y(x,3) = -8t^3 e^{-3x}$$

The solution in a series form is given by

$$y(x,t) = 6e^{-3x} - 12te^{-3x} + 12t^2 e^{-3x} + \ldots\ldots = 6e^{-3x} e^{-2t} = 6e^{-(3x+2t)}$$

Example 2.2:

Consider the following linear second order partial differential equation

$$u_{xx} + u_{tt} = 0,\quad u(x,0) = 0,\quad u_t(x,0) = \cos x \qquad (10)$$

To find the solution we take Elzaki transform of equation (10) and making use of the conditions to find,

$$E\left[u(x,t)\right] = v^3 \cos x - v^2 E\left[A_m\right],\qquad A_m = \frac{\partial^2 u(x,m)}{\partial x^2}$$

Take the inverse Elzaki transform we get:

$$u(x,m+1) = -E^{-1}\left\{v^2 E\left[A_m\right]\right\},\quad u(x,0) = 0,\quad u(x,1) = t\cos x \qquad (11)$$

By using equation (11), we find that:

$$A_1 = -t\cos x \Rightarrow u(x,2) = E^{-1}\left\{v^2 E\left[t\cos x\right]\right\} = \frac{t^3}{3!}\cos x$$

$$A_2 = -\frac{t^3}{3!}\cos x \Rightarrow u(x,3) = E^{-1}\left\{v^2 E\left[\frac{t^3}{3!}\cos x\right]\right\} = \frac{t^5}{5!}\cos x$$

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Then the solution is $u(x,t)=\cos x\left[t+\frac{t^3}{3!}+\frac{t^5}{5!}+.....\right]=\cos x\ \sinh t$

Example 2.3:

Consider the following second order nonlinear partial differential equation

$$\frac{\partial u}{\partial t}=\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\quad,\quad u(x.0)=x^2 \tag{12}$$

To find the solution take Elzaki transform of (12) and using the condition we get:

$$E\left[u(x,t)\right]=v^2x^2+vE\left[A_m+B_m\right]$$

Where $A_m=\sum_{m=0}^{h}\frac{\partial u(x,m)}{\partial x}\frac{\partial u(x,h-m)}{\partial x}$, $B_m=\sum_{m=0}^{h}u(x,m)\frac{\partial^2 u(h-m)}{\partial x^2}$, are projected differential transform of $\left(\frac{\partial u}{\partial x}\right)^2$, $u\frac{\partial^2 u}{\partial x^2}$

Take the inverse Elzaki transform to get:

$$u(x,m+1)=E^{-1}\left\{vE\left[A_m+B_m\right]\right\}\quad,\quad u(x,0)=x^2 \tag{13}$$

From equation (13) we find that:

$$A_0=4x^2\ ,\ B_0=2x^2\ \Rightarrow\ u(x,1)=E^{-1}\left[6x^2v^3\right]=6x^2t$$
$$A_1=48x^2t\ ,\ B_1=24x^2t\ \Rightarrow\ u(x,2)=E^{-1}\left[72x^2v^4\right]=36x^2t^2$$

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Then the solution of equation (12) is

$$u(x,t)=x^2+6x^2t+36x^2t^2+.....=x^2(1-6t)^{-1}=\frac{x^2}{1-6t}$$

Which is an exactly the same solution obtained by the Adomian decomposition method.

Example 2.4:

Let us consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t}=2u\left(\frac{\partial u}{\partial x}\right)^2+u^2\frac{\partial^2 u}{\partial x^2}\quad,\quad u(x,0)=\frac{x+1}{2} \tag{14}$$

Taking Elzaki transform of equation (14) and using the condition leads to:

$$E\left[u(x,t)\right]=v^{2}\left(\frac{x+1}{2}\right)+vE\left[2A_m+B_m\right]$$

Taking the inverse Elzaki transform to find:

$$u(x,m+1)=E^{-1}\{vE\left[2A_m+B_m\right]\} \quad ,u(x,0)=\frac{x+1}{2} \tag{15}$$

Where

$$A_m=\sum_{k=0}^{h}\sum_{m=0}^{k}u(x,m)\frac{\partial u(x,h-m)}{\partial x}\frac{\partial u(x,h-k)}{\partial x}$$

$$B_m=\sum_{k=0}^{h}\sum_{m=0}^{k}u(x,m)u(x,h-m)\frac{\partial^2 u(x,h-k)}{\partial x^2}$$

From equation (15) we have:

$$u(x,0)=\frac{x+1}{2}, \quad u(x,1)=\frac{x+1}{2}t, \quad u(x,2)=\frac{3(x+1)}{8}t^2, \ldots\ldots\ldots\ldots$$

Then the solution of equation (14) is

$$u(x,t)=\frac{x+1}{2}(1-t)^{-\frac{1}{2}}=\frac{x+1}{2\sqrt{1-t}}$$

3. Conclusion

The solution of linear and nonlinear partial differential equations can be obtained using Elzaki transform and projected differential transform method without any discretization of the variables. The results for all examples can be obtained in series form, and all calculations in the method are very easy. This technique is useful to solve linear and nonlinear partial differential equations.

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Table 1. Elzaki transform of some Functions

$f(t)$	$E[f(t)]=T(u)$
1	v^2
t	v^3
t^n	$n!\,v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sinat$	$\frac{av^3}{1+a^2v^2}$
$cosat$	$\frac{v^2}{1+a^2v^2}$